Math 111 – Tues, 5/17

- Questions on 4.6
- New material: 4.7 + 4.8

please turn in Mission 3

Week 8

Saturday 5/21 is the last day to change grading options

Thursday – Big Boss #2
Solutions to the review in my PCC

Part 1 – no calc

Part 2 – with calc

Q's on 4.6

17. $\log x + \log(x+15) = 2$

$\log(x(x+15)) = 2$

get a single log using properties

$\log_{10}(x^2+15x) = 2$

$10^2 = x^2+15x$

$0 = x^2+15x-100$

$0 = (x+20)(x-5)$

$x+20=0$ or $x-5=0$

$x = \cancel{-20}$ or 5

check domain:

$x > 0$ $\quad x+15>0$

$x>-15$

$\{\cancel{-20}, 5\}$ $\{5\}$

19. $\log(2x+1) = 1 + \log(x-2)$

$-\log(x-2) \qquad -\log(x-2)$

$\log(2x+1) - \log(x-2) = 1$

$\log_{10}\left(\frac{2x+1}{x-2}\right) = 1$

Check Domain:

$2x+1>0 \quad x-2>0$

$2x>-1 \quad x>2$

$x>-\frac{1}{2}$

$10^1 = \frac{2x+1}{x-2}$

$(x-2)10 = \frac{2x+1}{\cancel{x-2}}\cancel{(x-2)}$

$10x-20 = 2x+1$

$8x-20=1$

$8x=21$

$x = \frac{21}{8}$

$\left\{\frac{21}{8}\right\}$ ✓

ex: $e^{2} = e^{x+1}$

$2 = x + 1$

$1 = x$

$\{1\}$

ex: $\log_5 3 = \dfrac{\log 3}{\log 5}$ or $\dfrac{\ln 3}{\ln 5}$

Math 111 Lecture Notes

Section 4.7: Compound Interest

This section has *a lot* of formulas. You do not have to memorize the formulas in this section—the two you will need to use are given below and will be provided on any exams.

Compound Interest Formula:
The amount A after t years due to a principal P invested at an annual interest rate r compounded n times per year is

decimal

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

Continuous Interest Formula:
The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is

$$A = Pe^{rt}$$

Example 1. You invest $3,000 into a bank account. For each interest rate below, write the general formula and compute the value of the investment after 7 years.

P

t

- 5% compounded quarterly

$n=4$

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$A = 3000\left(1+\frac{.05}{4}\right)^{4\cdot 7}$$

$$\approx \$4,247.98$$

- 5% compounded monthly

$$A = 3000\left(1+\frac{.05}{12}\right)^{12\cdot 7}$$

$$\approx \$4,254.11$$

- 5% compounded daily

$$A = 3000\left(1+\frac{.05}{365}\right)^{365\cdot 7}$$

$$= 3000\left(1+\frac{.05}{365}\right)^{2555}$$

$$\approx \$4,257.10$$

Example 2. Complete Table 1 using the previous examples.

TABLE 1

$\left(1+\frac{r}{n}\right)^n$

Compounding Frequency	Annual Growth Factor	r Effective Annual Rate
Annual	1.05	5%
Quarterly	$\left(1+\frac{.05}{4}\right)^4 \approx 1.050945$	5.0945%
Monthly	$\left(1+\frac{.05}{12}\right)^{12} \approx 1.051162$	5.1162%
Daily	$\left(1+\frac{.05}{365}\right)^{365} \approx 1.051267$	5.1267%
Continuously	$e^{.05} \approx 1.0512711$	5.1271%

Pe^{rt}

Example 3. Now assume that you have \$1 and it earns 100% annual interest. Table 2 shows the growth factor for each of the compounding frequencies listed. (This is utterly silly in reality–but will show you exactly where e comes from!!)

TABLE 2

Compounding Frequency	Annual Growth Factor
Annual	$\left(1+\frac{1}{1}\right)^1 = 2$
Semi-annual	$\left(1+\frac{1}{2}\right)^2 \approx 2.25$
Quarterly	$\left(1+\frac{1}{4}\right)^4 \approx 2.441406$
Monthly	$\left(1+\frac{1}{12}\right)^{12} \approx 2.613035$
Daily	$\left(1+\frac{1}{365}\right)^{365} \approx 2.714567$
Hourly	$\left(1+\frac{1}{8760}\right)^{8760} \approx 2.718127$
Each minute	$\left(1+\frac{1}{525600}\right)^{525600} \approx 2.718279$
Each second	$\left(1+\frac{1}{31536000}\right)^{31536000} \approx 2.718282$
Continuously	$e^1 \approx 2.718282\ldots\ldots$

irrational # like π

$.0225$
r

Example 4. You invest \$5,000 into an account that earns 2.25% interest $\boxed{\text{compounded continuously.}}$

(a) Write the formula that models the value of this investment after t years.

$$A = Pe^{rt}$$
$$A = 5{,}000e^{.0225t}$$

(b) What will the value of the account be after 5 years?

$$A = 5{,}000e^{.0225(5)}$$
$$\approx \$5{,}595.36$$

(c) How long will it take for the account value to double?

$$\frac{10{,}000}{5{,}000} = \frac{5{,}000e^{.0225 \cdot t}}{5{,}000}$$

doubling $\rightarrow$ $2 = e^{.0225t}$

$$\ln 2 = \ln e^{.0225t}$$
$$\ln 2 = .0225t \underbrace{\boxed{\ln_e e}}_{=1}$$
$$\frac{\ln 2}{.0225} = \frac{.0225t}{.0225}$$
$$\frac{\ln 2}{.0225} = t$$
$$30.8 \text{ years} \approx t$$

or switch to log form

$$\ln(2) = .0225t$$

Effective Rate:

$r_{eff} = \left(1+\frac{r}{n}\right)^n - 1$ $\quad$ $r_{eff} = e^r - 1$

The **effective rate of interest** is the equivalent annual simple interest that would yield the same amount as compounding n times per year, or continuously, after 1 year.

Example 5. Determine which of the following interest rates for an investment is a better deal:

- 6% compounded monthly

$r_{eff} = \left(1 + \frac{.06}{12}\right)^{12} - 1$

$\approx .061678$

$\approx 6.1678\%$

- 5.95% compounded continuously

$r_{eff} = e^{.0595} - 1$

$\approx .061306$

$\approx 6.1306\%$

Group Work 1. Determine which of the following interest rates for an investment is a better deal:

- 9% compounded quarterly

$r_{eff} = \left(1 + \frac{.09}{4}\right)^{4} - 1$

$\approx .093083$

$\approx 9.3083\%$

- 8.95% compounded continuously e

$r_{eff} = e^{.0895} - 1$

$\approx .093627$

$\approx 9.3627\%$

Example 6. What interest rate (compounded continuously) is required for the value of an investment to double in 15 years?

t

$$A = Pe^{rt}$$

$$\frac{2\cancel{P}}{\cancel{P}} = \frac{\cancel{P}e^{r\cdot 15}}{\cancel{P}}$$

$$2 = e^{15r}$$

log form

$$\frac{\ln 2}{15} = \frac{15r}{15}$$

$$\frac{\ln 2}{15} = r$$

$$.0462 \approx r$$

A rate of 4.62% is needed for an investment to double in 15 years. (compounded continuously)

Example 7. What interest rate (compounded annually) is required for the value of an investment to triple in 15 years?

n t

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$\frac{3\cancel{P}}{\cancel{P}} = \frac{\cancel{P}}{\cancel{P}}\left(1+\frac{r}{1}\right)^{1\cdot t}$$

$$3 = (1+r)^t$$

$$\sqrt[15]{3} = \sqrt[15]{(1+r)^{15}}$$

inverse function

$$\sqrt[15]{3} = 1+r$$

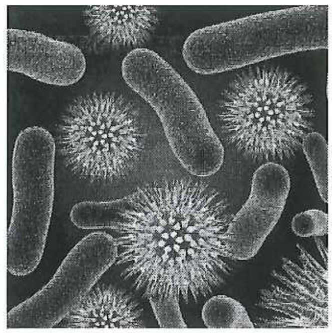
$$\sqrt[15]{3} - 1 = r$$

$\boxed{3^{1/15}-1}$

$$.07599 \approx r \quad \text{or } 7.6\%$$

Math 111 Lecture Notes

Section 4.8: Exponential Growth and Decay Models



Populations that obey **uninhibited growth** grow exponentially according to the formula

$$A(t) = \underline{A_0} e^{kt}$$

like $A = Pe^{rt}$

where k is the continuous growth rate and A_0 is the initial amount.

Substances that undergo **uninhibited radioactive decay** do so exponentially according to the formula

$$N(t) = \underline{N_0} e^{kt}$$

where k is the continuous decay rate and N_0 is the initial amount.

The **doubling time** for a population is the amount of time it takes a population growing exponentially to double in size.

The **half-life** for a radioactive substance is the amount of time it takes for the quantity of the substance to be one half its original amount.

$A_0 = 8$

Example 1. The fruit fly drosphilia have a doubling time of 10 days.[2] There are initially 8 fruit flies.

(a) The population of fruit flies is modeled by $\cancel{N}(t) = \cancel{N_0}e^{kt}$. Use the doubling time to find the value of k.

$A(t) = A_0 e^{kt}$

$$\frac{16}{8} = \frac{8e^{k(10)}}{8}$$

$$2 = e^{10k}$$

$$\ln 2 = \ln e^{10k}$$

$$\frac{\ln 2}{10} = \frac{10k}{10}$$

$$k = \frac{\ln 2}{10} \approx .0693$$

(b) What is the continuous growth rate? k

$k = .0693$ or 6.93%

(c) Write the full formula for $\cancel{N(t)}$. $A(t)$

$$A(t) = 8e^{.0693t}$$

(d) How many fruit flies will there be after 30 days?

$$A(30) = 8e^{.0693(30)}$$

$$\approx 63.97$$

There will be about 64 fruit flies.

(e) When will there be 1000 fruit flies?

$$\frac{1000}{8} = \frac{8e^{.0693t}}{8}$$

$$125 = e^{.0693t}$$

$$\ln 125 = \ln e^{.0693t}$$

$$\frac{\ln 125}{.0693} = \frac{.0693t}{.0693}$$

$t \approx 69.67$

After about 70 days.

[2] https://www.lscore.ucla.edu/hhmi/performance/VickiHahmFinal.pdf

Example 2. The half-life of carbon-14 is $\boxed{\text{5600 years.}}$ Write the percentage of carbon-14, $\cancel{A}$ $N(t)$, remaining after t years of decay. Round the value you find for k accurate to six decimal places.

$$N(t) = N_0 e^{kt}$$

$$\frac{\frac{1}{2}\cancel{N_0}}{\cancel{N_0}} = \frac{\cancel{N_0}e^{k(5600)}}{\cancel{N_0}}$$

$\frac{1}{2}$ for half-life

$$\frac{1}{2} = e^{5600k}$$

$$\ln\left(\frac{1}{2}\right) = \ln e^{5600k}$$

$$\frac{\ln\left(\frac{1}{2}\right)}{5600} = \frac{5600k}{5600}$$

$$-.000124 \approx k$$

$$\boxed{N(t) = N_0 e^{-.000124t}}$$

Example 3. In 1991, two hikers discovered a historic iceman in the Ötztal Alps in Italy.[3] Assuming 46% of his carbon-14 was found remaining in the sample, how many years ago did the iceman die? Use the formula you found in the previous example.

$$N(t) = N_0 e^{-.000124t}$$

$$\frac{.46\cancel{N_0}}{\cancel{N_0}} = \frac{\cancel{N_0}e^{-.000124t}}{\cancel{N_0}}$$

$$.46 = e^{-.000124t}$$

$$\ln .46 = \ln e^{-.000124t}$$

$$\frac{\ln .46}{-.000124} = \frac{-.000124t}{-.000124}$$

$$6,262.3 \approx t$$

The iceman died about 6,262 years ago.

[3]http://www.nupecc.org/iai2001/report/B44.pdf

Example 4. The radioisotope Sodium-24 decays at a continuous rate of about 4.5% per hour. What is the half-life of this radioactive substance?[4]

$k = -.045$

t

$A(t) = A_0 e^{kt}$

$\frac{1}{2} = e^{-.045t}$

$\frac{\ln\left(\frac{1}{2}\right)}{-.045} = \frac{-.045t}{-.045}$

$15.4 \approx t$

hours

Example 5. The radioisotope Barium-139 has a half-life of 86 minutes. Find the continuous rate of decay.

$N(t) = N_0 e^{kt}$

$\frac{\frac{1}{2}\cancel{N_0}}{N_0} = \frac{\cancel{N_0}}{N_0} e^{k(86)}$

$\frac{1}{2} = e^{86k}$

$\frac{\ln \frac{1}{2}}{86} = \frac{86k}{86}$

Should be negative for decay $\rightarrow$ $-.0081 \approx k$

The continuous rate of decay is about 0.81% per minute.

[4] http://www.ndt-ed.org/EducationResources/HighSchool/Radiography/halflife2.htm

Example 6. The half-life of Cobalt-60 is 5.27 years.[5]. If 15 grams are present now, how many grams will be present in 100 years? N_0

2 Steps: ① use the half-life to find k and write the formula

② use the formula with k to answer the question

① $\frac{1}{2} = e^{k(5.27)}$

$\frac{\ln \frac{1}{2}}{5.27} = \frac{5.27k}{5.27}$

$-.1315 \approx k$

$N(t) = 15e^{-.1315t}$

② $N(100) = 15e^{-.1315(100)}$

$\approx .000029$

After 100 years, only .000029 grams remain.

[5] http://www.bt.cdc.gov/radiation/isotopes/cobalt.asp