Math 111 - Wed, May 4th

Questions on 4.3 + 4.4

→ Checkpoint 5

Finish 4.4 - page 6 + supplement

New material : 4.5

Mission 3 + Bonus due Monday

Project due next week

(no office hours tomorrow - you can email me)

Math 111 Lecture Notes

Section 4.5: Properties of Logarithms

Example 1. Calculate the following:

(a) $\log_5(1) = 0$ because $5^0 = 1$

(c) $\log_2(1) = 0$

(e) $\log(1) = 0$

(g) $\ln(1) = 0$

(b) $\log_5(5) = 1$

(d) $\log_2(2) = 1$

(f) $\log_{10}(10) = 1$

(h) $\ln(e) = 1$

For any positive real number a, $a \neq 1$, it holds that

- $\log_a(1) = 0$
- $\log_a(a) = 1$

Example 2. We have said that the functions defined by $g(x) = \log_2(x)$ and $f(x) = 2^x$ are inverse functions. Find $f(g(x))$ and $g(f(x))$. Since f and g are inverses, what should these be equivalent to?

$f(g(x)) = f(\log_2 x)$

$= 2^{\log_2 x}$ because they are inverses

$= x$

$g(f(x)) = g(2^x)$

$= \log_2(2^x)$

$= x$

ex: $10^{\log_{10} 17} = 17$

$\log_{10} 10^{17} = 17$

For any positive real numbers x and a, $a \neq 1$, it holds that

- $\log_a(a^x) = x$
- $a^{\log_a(x)} = x$

Example 3. Compare the following expressions:

$\log_2(8) + \log_2(4)$ vs. $\log_2(32)$

$3 + 2$

5

5

Example 4. Compare the following expressions:

$\log_3(81) - \log_3(3)$ vs. $\log_3(27)$

$3^4 = 81$

$4 - 1$

3

3

Example 5. Compare the following expressions:

$4\log(10)$ $= \log 10^4$ vs. $\log(10000)$

$4 \cdot 1$

$= 4$

4

For any positive real numbers M, N, and a, $a \neq 1$, it holds that

- $\log_a(MN) = \log_a(M) + \log_a(N)$
- $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$
- $\log_a(M^r) = r\log_a(M)$

Example 6. Use the properties of logarithms to find the exact value of the following expressions. Do not use a calculator.

(a) $\log_4\left(4^{-5}\right)$ $= -5$

property 3

(d) $2^{\log_2(15)}$ $= 15$

property 4

(b) $\log_6(9) + \log_6(4)$

property 5 $= \log_6(36)$

$= 2$

(e) $\log(250) - \log(25)$

$\log\left(\frac{250}{25}\right)$ ↙ property 6

$= \log_{10} 10$ ↙ property 2

$= 1$

(c) $e^{\ln(7)}$ $= 7$

property 3

(f) $5^{\log_5(6)+\log_5(7)}$

$5^{\log_5(42)}$ prop 5

$= 42$ prop 4

Example 7. Write each expression as a sum and/or difference of logarithms. Express powers as factors.

(a) $\log\left(\frac{1}{x-3}\right), x > 3$

$\frac{1}{x-3} > 0$ Domain of Log is $(0, \infty)$

$x - 3 > 0$

$= \log 1 - \log(x-3)$ prop 6

$= 0 - \log(x-3)$

$= -\log(x-3), \quad x > 3$

(b) $\ln\left(x^4\sqrt{1+x^2}\right)$

$= \ln x^4 + \ln(\sqrt{1+x^2})$ prop 5

$= 4\ln x + \ln\sqrt{1+x^2}$ prop 7

$= 4\ln x + \ln(1+x^2)^{\frac{1}{2}}$ prop 7

$= 4\ln x + \frac{1}{2}\ln(1+x^2)$

(c) $\log_5\left(\frac{\sqrt[3]{x^2+1}}{x^2-1}\right)$

$= \log_5(\sqrt[3]{x^2+1}) - \log_5(x^2-1)$ prop 6

$= \log_5(x^2+1)^{1/3} - \log_5(x^2-1)$

$= \frac{1}{3}\log_5(x^2+1) - \log_5((x-1)(x+1))$ prop 7

$= \frac{1}{3}\log_5(x^2+1) - [\log_5(x-1) + \log_5(x+1)]$ prop 5

$= \frac{1}{3}\log_5(x^2+1) - \log_5(x-1) - \log_5(x+1)$

Example 8. Write each expression as a single logarithm.

(a) $\log_2\left(\frac{x-3}{x+5}\right)+\log_2\left(\frac{3x+15}{x-4}\right)$ $= \log_2\left(\frac{x-3}{x+5}\cdot\frac{3x+15}{x-4}\right)$ prop 5

$= \log_2\left(\frac{x-3}{x+5}\cdot\frac{3(x+5)}{x-4}\right)$

$= \log_2\left(\frac{3(x-3)}{x-4}\right)$

(b) $\log_4\left(\frac{5}{x}\right)-\log_4\left(\frac{x+2}{x^3}\right)$ $= \log_4\left(\frac{5}{x}\cdot\frac{x^3}{x+2}\right)$ prop 6

$= \log_4\left(\frac{5x^2}{x+2}\right)$

(c) $\log\left(x^2+3x+2\right)-2\log\left(x+1\right)$ $= \log[(x+2)(x+1)]-\log(x+1)^2$ prop 7

$= \log\left(\frac{(x+2)(x+1)}{(x+1)(x+1)}\right)$ prop 6

$= \log\left(\frac{x+2}{x+1}\right)$

(d) $\ln(x^2-9)+\ln\left(\frac{x}{x-3}\right)-\ln\left(\frac{x+3}{x}\right)$ $= \ln(x+3)(x-3)+\ln\left(\frac{x}{x-3}\right)-\ln\left(\frac{x+3}{x}\right)$

$= \ln\left(\frac{x(x+3)(x-3)}{x-3}\right)-\ln\left(\frac{x+3}{x}\right)$ prop 5

$= \ln(x(x+3))-\ln\left(\frac{x+3}{x}\right)$

$= \ln\left(\frac{x(x+3)}{1}\cdot\frac{x}{x+3}\right)$ prop 6

$= \ln x^2$

$= 2\ln x$ prop 7

Change-of-Base Formula If a, b, and M are positive real numbers, $a \neq 1$ and $b \neq 1$, then

$$\log_a(M) = \frac{\log_b(M)}{\log_b(a)} \qquad \log_3 17 = \frac{\log 17}{\log 3}$$

In practice, we primarily use one of the following forms of this formula:

$$\log_a(M) = \frac{\log(M)}{\log(a)} \quad \text{or} \quad \log_a(M) = \frac{\ln(M)}{\ln(a)}$$

or $\frac{\ln 17}{\ln 3}$

Example 9. Use the Change-of-Base formula to write the following logarithmic expressions in terms of the natural logarithmic function or common logarithmic function. Then approximate each in your calculator.

(a) $\log_4(15) = \frac{\ln 15}{\ln 4}$ prop 8 ≈ 1.9535

(b) $\log_5\left(\frac{1}{7}\right) = \frac{\ln(1/7)}{\ln 5} = -1.2091$

Properties of Logarithms

$$y = \log_a(x) \text{ if and only if } x = a^y$$

If a, b, M and N are positive real numbers, $a \neq 1$ and $b \neq 1$, then

(1) $\log_a(1) = 0$

(2) $\log_a(a) = 1$

(3) $\log_a\left(a^x\right) = x$

(4) $a^{\log_a(x)} = x$

(5) $\log_a(MN) = \log_a(M) + \log_a(N)$

(6) $\log_a\left(\dfrac{M}{N}\right) = \log_a(M) - \log_a(N)$

(7) $\log_a\left(M^r\right) = r\log_a(M)$

(8) $\log_a(M) = \dfrac{\log_b(M)}{\log_b(a)}$

$\ln(x) = \log_e(x)$ $\qquad$ $\log(x) = \log_{10}(x)$