Chapter D: Democracy

Student Outcomes for this Chapter

Section D1: Voting Methods

Students will be able to:

- Read a voter preference schedule for ranked choice voting
- Calculate the minimum number of votes to win a majority and plurality
- Find the winner of an election using the plurality method
- Find the winner of an election using the instant runoff method
- Find the winner of an election using the Borda count method
- Find the winner of an election using the pairwise (Condorcet) method

Section D2: Popular Vote, Electoral College and Voting Power

Students will be able to:

- Calculate the number of electors per state
- Determine the winner of the popular vote
- Determine the winner of the electoral college
- Calculate the electoral power of each state

Section D3: Apportionment and the 3/5 Compromise

Students will be able to:

- Apportion representatives using Hamilton's method
- Apportion representatives using Jefferson's method
- Apportion representatives using Webster's method
- Apportion representatives using the Hill-Huntington Method
- 3/5 Compromise

Section D4: Gerrymandering and the Efficiency Gap

Students will be able to:

- Calculate the overall percentage of voters in each party
- Calculate the percentage of representative seats held by each party
- Calculate the efficiency gap for a given map
- Calculate the percentage that each seat is worth
- Determine the number of seats that the efficiency gap represents

Section D1 Voting Methods

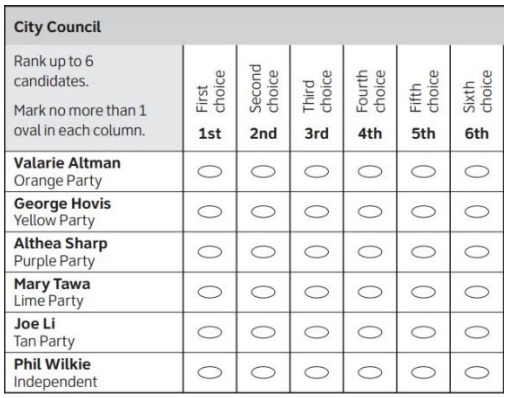
Voting

In many decision-making situations, it is necessary to gather the group consensus. This happens when a group of students decides where to hold a meeting, when a company decides which advertising slogan to use, and when a democratic country elects its leaders.

You may be most familiar with systems where you get one vote and the candidate with the most votes wins. However, there are many ways of determining a winner. We will look at several methods in this section.

Ranked Choice Ballots

To use some of the methods we are going to study, we need to know more than just each person's first choice. We are going to collect a ranked choice ballot. This type of ballot has each person rank all the options in order of their preference. The image to the right shows an example of a ranked choice ballot.



City Council

Rank up to 6 candidates.

Mark no more than 1 oval in each column.

	First choice 1st	Second choice 2nd	Third choice 3rd	Fourth choice 4th	Fifth choice 5th	Sixth choice 6th
Valarie Altman Orange Party	⬭	⬭	⬭	⬭	⬭	⬭
George Hovis Yellow Party	⬭	⬭	⬭	⬭	⬭	⬭
Althea Sharp Purple Party	⬭	⬭	⬭	⬭	⬭	⬭
Mary Tawa Lime Party	⬭	⬭	⬭	⬭	⬭	⬭
Joe Li Tan Party	⬭	⬭	⬭	⬭	⬭	⬭
Phil Wilkie Independent	⬭	⬭	⬭	⬭	⬭	⬭

Example 1: Students are voting for their class president and the candidates are Omar (O), Ana (A), and Helena (H). They have ranked the candidates according to their preference.

	Vien	Ana	Marv	Tasha	Eve	Omar	Lupe	Dave	Helena	Jimmy
1st choice	A	A	O	H	A	O	H	O	H	A
2nd choice	O	H	H	A	H	H	A	H	A	H
3rd choice	H	O	A	O	O	A	O	A	O	O

Preference Schedule

To simplify this data, we can combine all the people who voted with the exact same ranking. For example, Tasha, Lupe, and Helena, all voted in the same way, HAO. So the number 3 is placed at the top of that column. Three students also voted for AHO, and OHA. One student voted for the ranking AOH. Below is the combined table, called a preference schedule or table.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Notice that by totaling the vote counts across the top of the preference schedule we can see the total number of votes cast: $1+3+3+3 = 10$ total votes.

Plurality Method

The voting method you may be most familiar with in the United States is the plurality method. In this method, the candidate with the most first-choice votes is declared the winner.

There is a difference between a majority and a plurality. For a majority, a candidate must have over 50% of the votes. There are some states that require a majority in order to win an election. In more cases, though, only a plurality is required, which means having more votes than any other candidate. If a tie occurs without using ranked choice voting, then a new run-off election would be required.

Example 2: In our election from above, we had this preference table:

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

For the plurality method, we only look at the 1st choice row. Totaling them up:

Ana: $1+3 = 4$ votes

Omar: 3 votes

Helena: 3 votes

Ana is the winner using the plurality voting method.

Notice that Ana won with 4 out of 10 votes, 40% of the votes, which is a plurality of the votes, but not a majority.

How Many Votes are Needed to Win

If we know how many candidates and voters there are, we can calculate the minimum number of votes needed for a majority win and a plurality win.

Continuing example 2 above, there were three candidates and 10 voters. To find the minimum number of votes needed for a plurality win, we divide

$10 \div 3 \approx 3.33$. Then we round up to 4 votes.

We cannot have a partial vote, and a candidate would need more than 1/3 of the vote as a minimum, so we round up to the next whole number. Four votes would be the least a candidate could win with.

Let's say there were only 2 candidates. Then to find the majority needed, we divide

$10 \div 2 = 5$. Then we round up to 6 votes.

Exactly half of the votes would be a tie, so we again round up to the next whole number. A candidate would need 6/10 to win a majority.

Insincere Voting

In a system with only two major political parties like the United States, it can be nearly impossible for third-party candidates to break through. You may feel like you would be wasting your vote if you voted for a third-party candidate. Consider this example.

Example 3: Here is a two-party election with preferences shown below. It is a close race, but candidate B would win.

Number of voters	96	100
1st choice	A	B
2nd choice	B	A

Suppose a third candidate, C, entered the race, and a segment of voters sincerely want to vote for that third candidate. Here is the new preference schedule.

Number of voters	96	90	10
1st choice	A	B	C
2nd choice	B	A	B
3rd choice	C	C	A

The ten people who prefer candidate C are in a bind because if they vote for C, then A will win, but they prefer B over A. It is likely that they will vote for B, their second choice, to keep A from winning.

Situations like the one above can lead to insincere voting. Insincere voting is when a person casts a ballot counter to their actual preference for strategic purposes. In the case above, a person who wants to vote for a third-party candidate realizes it would take a vote away from their major party candidate. Not wanting to see their party lose the election, they vote for the major party candidate.

Instant Runoff Method

The Instant Runoff Method is a modification of the plurality method that attempts to address the issue of insincere voting.

The voting is done with ranked choice ballots and a preference schedule is generated. Then we tally all the first-choice votes. The candidate with the *least* first-place votes is eliminated, but any votes for that candidate are transferred to the voters' next choice.

The process continues until one candidate has a majority. We will show you how it works with an example.

Example 4: Consider the preference schedule below, in which a committee is choosing a chairperson. The candidates are labeled A, B, C, D, and E for simplicity. Here is the preference schedule:

	3	5	1	5	7	2	1
1st choice	B	C	C	B	D	A	E
2nd choice	C	A	B	D	C	E	D
3rd choice	A	E	B	C	A	B	B
4th choice	D	D	E	A	E	C	A
5th choice	E	B	A	E	B	D	C

Let's start by tallying the first-choice votes:

A: 2 B: 8 C: 6 D: 7 E: 1

If this was a plurality election, note that B would be the winner with 8 first-choice votes, compared to 7 for D, 6 for C, 2 for A, and 1 for E.

Round 1: We make our first elimination. Choice E has the fewest first-place votes, so we eliminate E from the election. To transfer the 1 vote, we look at the person who voted for E. Their second choice was D, so we transfer that one vote to D, who will now have 8 votes:

A: 2 B: 8 C: 6 D: 7 ~~E: 1~~
+1
8

Round 2: There are 24 votes in total, so a majority would be 13 votes. Since no candidate has the majority, we eliminate the next candidate with the fewest votes, which is A. We look at the column for the two people who chose A. Their second choice was E, who is already eliminated. Their third choice is B, so their 2 votes are transferred to B, for a total of 10:

~~A: 2~~ B: 8 C: 6 D: 7 ~~E: 1~~
+2 +1
10 8

Round 3: Now C has the fewest votes, and the there are two columns for C in the preference schedule. Five people who voted for C chose A second, then E (who have already been eliminated), then D. So we transfer their 5 votes to D. One person who voted for C selected B as their second choice, so one vote goes to B. Then we tally the votes for each remaining candidate.

~~A: 2~~ B: 8 ~~C: 6~~ D: 7 ~~E: 1~~
+2 +1

10	8
+1	+5
11	13

Now we can see that D is the winner of the instant runoff method with 13 votes. We can stop this method earlier if a candidate reaches a majority.

The Instant Runoff Method is similar to the idea of holding runoff elections, but since every voter's order of preference is recorded on the ballot, the runoff can be computed quickly without requiring a second costly election. It also makes it safe to vote for a third-party candidate knowing that your vote will go to your second choice if your first choice can't win.

This voting method is used in several places around the world, including the election of members of the Australian House of Representatives, statewide elections in Maine, and local elections in Benton County, Oregon[1]. A version of the Instant Runoff Method is used by the International Olympic Committee to select host nations.

Borda Count (Point System)

Borda Count is another voting method, named for Jean-Charles de Borda, who developed the system in 1770.

In this method, points are assigned to candidates based on their ranking; 1 point for last choice, 2 points for second-to-last choice, and so on. The point values for all ballots are totaled, and the candidate with the largest point total is the winner.

Example 5: A group of PCC students are getting together for a student leadership conference. Their members are from four campuses: Sylvania, Rock Creek, Cascade and Southeast. The votes for where to hold the conference were:

	31	25	10	14
1st choice	Sylvania	Rock Creek	Southeast	Cascade
2nd choice	Southeast	Cascade	Cascade	Rock Creek
3rd choice	Rock Creek	Sylvania	Sylvania	Southeast
4th choice	Cascade	Southeast	Rock Creek	Sylvania

Now we will add a column on the left side of the table for the points. In each of the 31 ballots represented by the first column, Cascade will be given 1 point, Rock Creek 2 points, Southeast 3 points, and Sylvania 4 points. Then we multiply the points by the number of voters and for each campus:

Points		31	25	10	14
4	1st choice	Sylvania	Rock Creek	Southeast	Cascade
3	2nd choice	Southeast	Cascade	Cascade	Rock Creek

[1] https://www.fairvote.org/rcv#where_is_ranked_choice_voting_used

2	3rd choice	Rock Creek	Sylvania	Sylvania	Southeast
1	4th choice	Cascade	Southeast	Rock Creek	Sylvania

Sylvania: $1\cdot 14+2\cdot 35+3\cdot 0+4\cdot 31=208$ points

Rock Creek: $1\cdot 10+2\cdot 31+3\cdot 14+4\cdot 25=214$ points

Cascade: $1\cdot 31+2\cdot 0+3\cdot 35+4\cdot 14=192$ points

Southeast: $1\cdot 25+2\cdot 14+3\cdot 31+4\cdot 10=186$ points

Using the Borda count method, Rock Creek is the winning location.

Borda count is sometimes described as a consensus-based voting system, since it can be used to choose a more broadly acceptable option over the one with first-choice support. This is a different approach than plurality and instant runoff voting that focus on first-choice votes; Borda Count considers every voter's entire ranking to determine the outcome.

Because of this consensus behavior, Borda Count, or some variation of it, is commonly used in awarding sports awards. Variations are used to determine the Most Valuable Player in baseball, to rank teams in NCAA sports, and to award the Heisman trophy.

Pairwise Comparisons (Copeland's or Condorcet Method)

Another method, called pairwise comparisons or the Condorcet method, attempts to be fair by looking at each pair of candidates separately.

In this method, we look at each pair as if they were the only two candidates running and determine which of the two is more preferred. In Copeland's method, the more preferred candidate is awarded 1 point. If there is a tie, each candidate is awarded ½ point. After all the pairwise comparisons are made, the candidate with the most points, and hence the most pairwise wins, is declared the winner.

Variations of Copeland's Method are used in many professional organizations, including election of the Board of Trustees for the Wikimedia Foundation that runs Wikipedia.

Example 6: Consider our class president example from the beginning of the chapter. Determine the winner using Copeland's Method.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

We need to look at each pair of candidates to see which would win in a one-to-one comparison. For example, comparing Helena with Omar, we see that 6 voters, those shaded in the table below, would prefer Helena to Omar. Note that

Helena doesn't have to be the voter's first choice – we're imagining that Ana wasn't in the race. The other 4 people chose Omar over Helena.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Based on this comparison of Helena vs. Omar, Helena wins 1 point.

Next, comparing Ana with Omar, the 1 voter in the first column prefers Ana, as do the 3 voters in the second column. The 3 voters in the third column prefer Omar, but the 3 voters in the last column would choose Ana. So, altogether 1+3+3=7 voters prefer Ana over Omar, and 3 prefer Omar over Ana. So, comparing Ana vs Omar: 7 votes to 3 votes: Ana gets 1 point.

To summarize, we'll list all combinations of the 3 candidates and their votes:

Helena-6 vs Omar-4 Helena gets 1 point

Ana-7 vs Omar-3 Ana gets 1 point

Helena-6 vs Ana-4 Helena gets 1 point

Ana gets 1 point and Helena gets 2 points. Helena is the winner under Copeland's Method, having earned the most points. This method gave us a different winner than the plurality method where Ana won, because more people preferred Helena to Ana, even though they didn't choose Helena for their first choice.

Now that we've seen an example of each voting method, let's run through each method for the same scenario.

Example 7: Consider an advertising team voting to choose one of four different slogans, labeled A, B, C and D. Determine the winner using each method we have learned. method.

	5	3	6	4	2
1st choice	D	A	C	B	A
2nd choice	A	C	B	D	D
3rd choice	C	B	A	A	C
4th choice	B	D	D	C	B

Plurality Method: We tally the first-place votes:

A: 5 B: 4 C: 6 D: 5 C wins the plurality method.

Instant Runoff Method: We start with the plurality tallies. We look for the slogan with the least first-choice votes, and that is B. Looking in the preference schedule

at the 4 people who voted for B, we see their second choice is D. So we transfer those 4 votes to D, which now has 9.

A: 5 ~~B: 4~~ C: 6 D: 5
+4
9

Now A has the least number of votes with 5, so we look at the voters who chose A for their first choice. There are two columns, so three of those votes will be transferred to C and 2 will be transferred to D.

~~A: 5~~ ~~B: 4~~ C: 6 D: 5
+4
9
+3 +2
9 11 D is the winner of the instant runoff method.

Borda Count: We add a column on the left for the points, starting with 1 point for last place and counting up. Then we multiply the points by the number of votes and add them all up.

Points		5	3	6	4	2
4	1st choice	D	A	C	B	A
3	2nd choice	A	C	B	D	D
2	3rd choice	C	B	A	A	C
1	4th choice	B	D	D	C	B

A: $1\cdot 0+2\cdot 10+3\cdot 5+4\cdot 5=$ 55 points

B: $1\cdot 7+2\cdot 3+3\cdot 6+4\cdot 4=47$ points

C: $1\cdot 4+2\cdot 7+3\cdot 3+4\cdot 6=51$ points

D: $1\cdot 9+2\cdot 0+3\cdot 6+4\cdot 5=47$ points A is the winner of the Borda count method.

Pairwise Comparison: First we list all of the possible pairs:

A vs B B vs C C vs D

A vs C B vs D

A vs D

Then we count the number of voters who would choose each candidate if they were the only two in the race.

A-10 vs B-10 B–4 vs C-16 C-9 vs D-11

A-14 vs C-6 B-13 vs D-7

A-11 vs D-9

Totaling one point for each win and half a point for each tie gives us:

A has 2 ½ points B has 1 ½ points

C has 1 points D has 1 point

A is the winner of the pairwise comparison method.

Which Method is the Most Fair?

To see a very simple example of how difficult voting can be, consider the election below:

	5	5	5
1st choice	A	C	B
2nd choice	B	A	C
3rd choice	C	B	A

Notice that in this election:

10 people prefer A to B

10 people prefer B to C

10 people prefer C to A

No matter whom we choose as the winner, 2/3 of voters would prefer someone else! This scenario is dubbed Condorcet's Voting Paradox. In this election, there is no fair resolution. There are many additional paradoxes that are explained in David Lippman's original version of Math in Society[2].

It is because of this impossibility of a totally fair method that Plurality, Instant Runoff, Borda Count, Copeland's Method, and dozens of variants are all still used. Usually the decision of which method to use is based on what seems most fair for the situation in which it is being applied.

[2] Math in Society: Voting Theory, by David Lippman

Exercises D1

1. To decide on a new website design, the designer asks people to rank three designs that have been created (labeled A, B, and C). The individual ballots are shown below. Create a preference table.

   ABC, ABC, ACB, BAC, BCA, BCA, ACB, CAB, CAB, BCA, ACB, ABC

2. To decide on a movie to watch, a group of friends all vote for one of the choices (labeled A, B, and C). The individual ballots are shown below. Create a preference table.

   CAB, CBA, BAC, BCA, CBA, ABC, ABC, CBA, BCA, CAB, CAB, BAC

3. The planning committee for a renewable energy trade show is trying to decide what city to hold their next show in. The votes are shown below.

Number of voters	9	19	11	8
1st choice	Buffalo	Atlanta	Chicago	Buffalo
2nd choice	Atlanta	Buffalo	Buffalo	Chicago
3rd choice	Chicago	Chicago	Atlanta	Atlanta

   a. How many voters voted in this election?
   b. How many votes are needed for a majority? A plurality?
   c. Find the winner under the plurality method.
   d. Find the winner under the Instant Runoff Voting method.
   e. Find the winner under the Borda Count Method.
   f. Find the winner under Copeland's method.

4. A non-profit agency is electing a new chair of the board. The votes are shown below.

Number of voters	11	5	10	3
1st choice	Atkins	Cortez	Burke	Atkins
2nd choice	Cortez	Burke	Cortez	Burke
3rd choice	Burke	Atkins	Atkins	Cortez

   a. How many voters voted in this election?
   b. How many votes are needed for a majority? A plurality?
   c. Find the winner under the plurality method.
   d. Find the winner under the Instant Runoff Voting method.
   e. Find the winner under the Borda Count Method.
   f. Find the winner under Copeland's method.

5. The student government is holding elections for president. There are four candidates (labeled A, B, C, and D for convenience). The preference schedule for the election is:

Number of voters	120	50	40	90	60	100
1st choice	C	B	D	A	A	D
2nd choice	D	C	A	C	D	B
3rd choice	B	A	B	B	C	A
4th choice	A	D	C	D	B	C

   a. How many voters voted in this election?
   b. How many votes are needed for a majority? A plurality?
   c. Find the winner under the plurality method.
   d. Find the winner under the Instant Runoff Voting method.
   e. Find the winner under the Borda Count Method.
   f. Find the winner under Copeland's method.

6. The homeowners association is deciding a new set of neighborhood standards for architecture, yard maintenance, etc. Four options have been proposed. The votes are:

Number of voters	8	9	11	7	7	5
1st choice	B	A	D	A	B	C
2nd choice	C	D	B	B	A	D
3rd choice	A	C	C	D	C	A
4th choice	D	B	A	C	D	B

   a. How many voters voted in this election?
   b. How many votes are needed for a majority? A plurality?
   c. Find the winner under the plurality method.
   d. Find the winner under the Instant Runoff Voting method.
   e. Find the winner under the Borda Count Method.
   f. Find the winner under Copeland's method.

7. Consider an election with 129 votes.
   a. If there are 4 candidates, what is the smallest number of votes that a plurality candidate could win with?
   b. If there are 8 candidates, what is the smallest number of votes that a plurality candidate could win with?
8. Consider an election with 953 votes.
   a. If there are 7 candidates, what is the smallest number of votes that a plurality candidate could win with?
   b. If there are 8 candidates, what is the smallest number of votes that a plurality candidate could win with?

Exploration Questions

9. The Coombs method is a variation of instant runoff voting. In Coombs method, the choice with the most last place votes is eliminated. Apply Coombs method to the preference schedules from questions 5 and 6.

10. Copeland's Method is designed to identify a Condorcet Candidate if there is one, and is considered a Condorcet Method. There are many Condorcet Methods, which vary primarily in how they deal with ties, which are very common when a Condorcet winner does not exist. Copeland's method does not have a tie-breaking procedure built-in. Research the Schulze method, another Condorcet method that is used by the Wikimedia foundation that runs Wikipedia, and give some examples of how it works.

11. Instant Runoff Voting and Approval voting have supporters advocating that they be adopted in the United States and elsewhere to decide elections. Research comparisons between the two methods describing the advantages and disadvantages of each in practice. Summarize the comparisons, and form your own opinion about whether either method should be adopted.

12. In a primary system, a first vote is held with multiple candidates. In some many states, where voters must declare a party to vote in the primary election, and they are only able to choose between candidates for their declared party. The top candidate from each party then advances to the general election. Compare and contrast this primary with general election system to instant runoff voting, considering both differences in the methods, and practical differences like cost, campaigning, fairness, etc.

13. Sometimes in a voting scenario it is desirable to rank the candidates, either to establish preference order between a set of choices, or because the election requires multiple winners. For example, a hiring committee may have 30 candidates apply, and need to select 6 to interview, so the voting by the committee would need to produce the top 6 candidates. Describe how Plurality, Instant Runoff Voting, Borda Count, and Copeland's Method could be extended to produce a ranked list of candidates.