Math 111 Lecture Notes

Section 3.4: Properties of Rational Functions

A **rational function** is of the form $R(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomial functions.

Basic Rational Functions

Figure 1. Graph of $y = \frac{1}{x}$

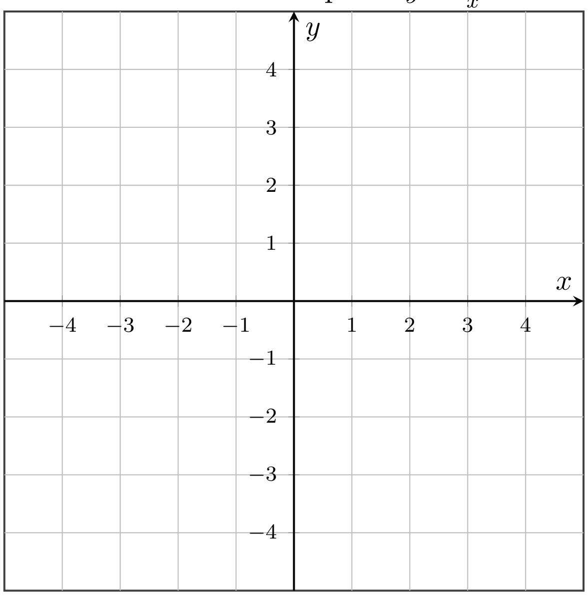


Table 1

x	$y = \frac{1}{x}$
-4	
-2	
-1	
-1/2	
-1/4	
0	
1/4	
1/2	
1	
2	
4	

Figure 2. Graph of $y = \frac{1}{x^2}$

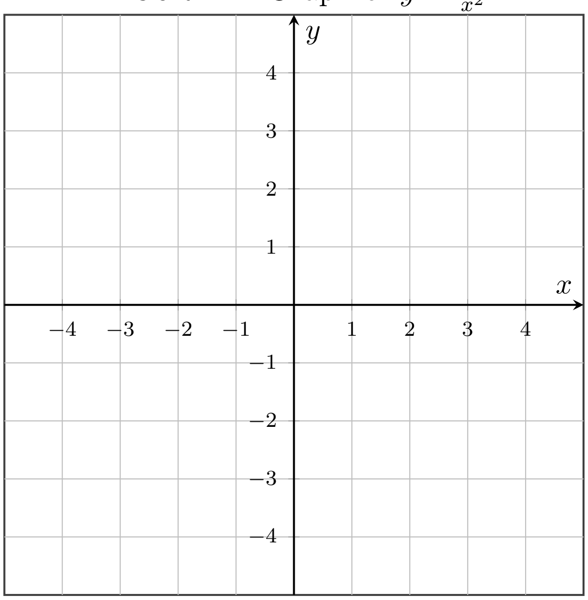


Table 2

x	$y = \frac{1}{x^2}$
-4	
-2	
-1	
-1/2	
-1/4	
0	
1/4	
1/2	
1	
2	
4	

Basic Rational Functions (close up)

Figure 3. Odd Powers

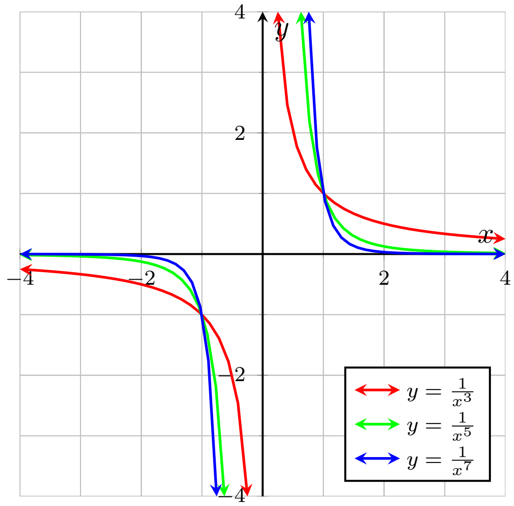
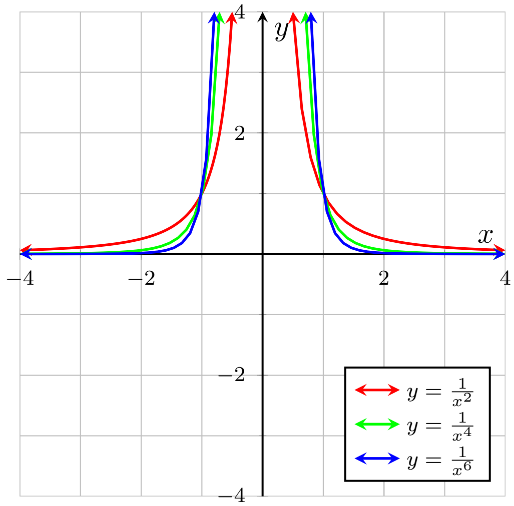


Figure 4. Even Powers



Example 1. Use transformations to graph the functions below. Clearly label any horizontal and vertical asymptotes.

(a) $R(x) = \dfrac{-2}{x}$

(b) $R(x) = \dfrac{1}{(x-1)^2} - 3$

Figure 5

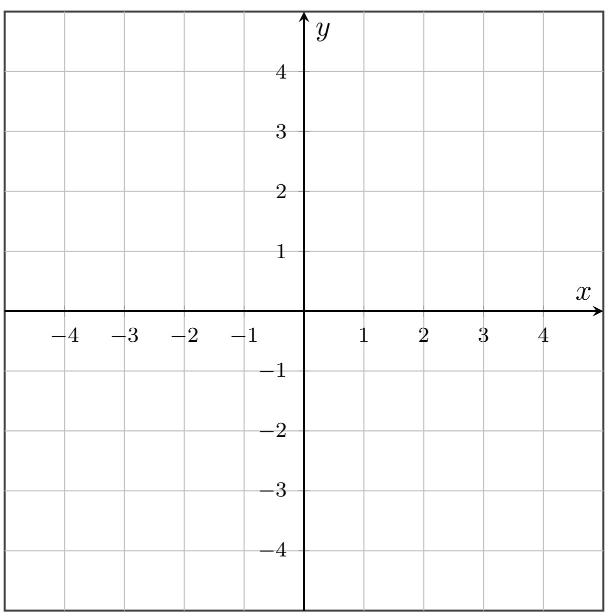
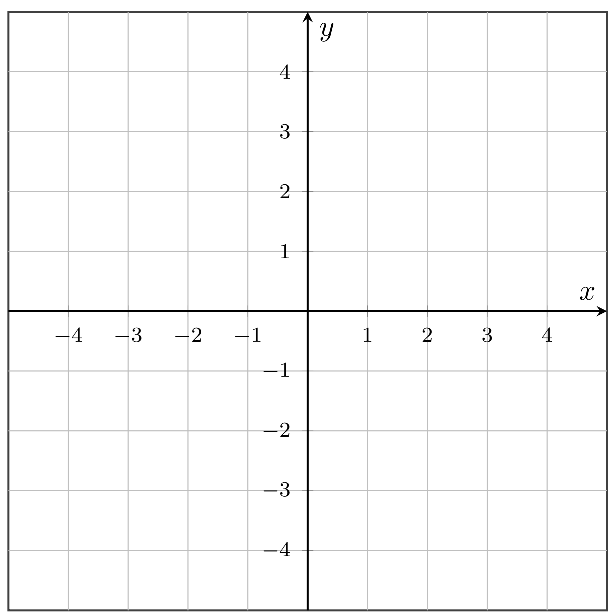


Figure 6



Determining the Horizontal Asymptote of a Rational Function

Let m be the degree of the function p in the numerator and let n be the degree of the function q in the denominator.

- If $m < n$, then the horizontal asymptote is $y = 0$.
- If $m = n$, then the horizontal asymptote is $y = c$ where c is a real number determined by the ratio of leading coefficients.
- If $m > n$, then no horizontal asymptote exists.

If $m = n + 1$, then an *oblique asymptote* exists.

Example 2. Determine any horizontal asymptotes for the following rational functions.

(a) $R(x) = \dfrac{5x+1}{10x^2+6x}$

(c) $R(x) = \dfrac{5x^3+1}{10x^2+6x}$

(b) $R(x) = \dfrac{5x^2+1}{10x^2+6x}$

(d) $R(x) = \dfrac{5x^4+1}{10x^2+6x}$

The **zeros** of a rational function are the values of x for which $p(x) = 0$, as the function's value is zero where the value of the numerator is zero. Most of the time, the zeros will occur at a when the factor $(x - a)$ is in the numerator of R.

A rational function is undefined where $q(x) = 0$, as this would cause division by zero.

A **vertical asymptote** occurs when the denominator of the *simplified* form of R is equal to zero. Most of the time, the vertical asymptote $x = b$ will occur when the factor $(x - b)$ is in the denominator of the *simplified* form of R.

A **hole** occurs when both the numerator and denominator equal zero for some value of x. We will identify a zero at c when the linear factor $(x - c)$ occurs in both the numerator and denominator of a rational function. Note that during simplification this factor cancels and results in a domain restriction for R.

Example 3. Find the any zeros, vertical asymptotes, and horizontal asymptotes for each rational function below.

(a) $R(x) = \dfrac{x-5}{x+6}$

(b) $R(x) = \dfrac{-10x}{5x-5}$

Example 4. Find the any zeros, holes, vertical asymptotes, and horizontal asymptotes for each rational function below. Factor each expression first and reduce to lowest terms if necessary.

(a) $R(x) = \dfrac{x^2 - 5x - 6}{x^2 + x - 12}$

(c) $R(x) = \dfrac{3}{x^3 - 4x}$

(b) $R(x) = \dfrac{3x - 6}{x^2 + x - 6}$

(d) $R(x) = \dfrac{2(x-1)(x+7)^2}{(x-1)(x+3)(x+4)}$

Group Work 1. Find the any zeros, holes, vertical asymptotes, and horizontal asymptotes for each rational function below. Factor each expression first and reduce to lowest terms if necessary.

(a) $R(x) = \dfrac{x^2 + 4x + 3}{2x^2 - 8}$

(b) $R(x) = \dfrac{8}{x^2 - 25}$

Group Work 2. Use transformations to graph the functions below. Clearly label any horizontal and vertical asymptotes.

(a) $R(x) = \dfrac{1}{x+2} - 1$

(b) $R(x) = -\dfrac{1}{x^2} + 3$

FIGURE 7

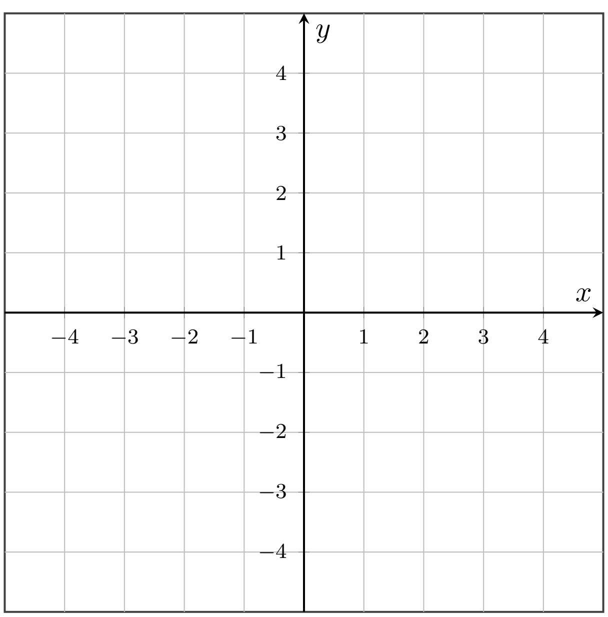


FIGURE 8