Math 111 - Wed, 5/25

Q's on 3.1
Checkpoint 7 (3.1)
Q's on 3.4
3.5

Checkpoint 8 next wed (3.4 + 3.5)
(No class Monday)
Class party - next wed - bring a snack to share if you wish

mission 4 due next wed

Final Boss - Mon, 6/6
regular class time
2 parts - no calc
calc

Final review + bonus given next wed.

Math 111 Lecture Notes

Section 3.5: Graphing Rational Functions

A **rational function** is of the form $R(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomial functions.

The **zeros** of a rational function are the values of x for which $p(x) = 0$, as the function's value is zero where the value of the numerator is zero. Most of the time, the zeros will occur at a when the factor $(x - a)$ is in the numerator of R.

A rational function is undefined where $q(x) = 0$, as this would cause division by zero. – domain

A **vertical asymptote** occurs when the denominator of the *simplified* form of R is equal to zero. Most of the time, the vertical asymptote $x = b$ will occur when the factor $(x - b)$ is in the denominator of the *simplified* form of R.

A **hole** occurs when both the numerator and denominator equal zero for some value of x. We will identify a zero at c when the linear factor $(x - c)$ occurs in both the numerator and denominator of a rational function. Note that during simplification this factor cancels and results in a domain restriction for R.

The **long run behavior** and **horizontal asymptote** of R can be determined by the ratio of leading terms of p and q.

ratio of leading terms

$\dfrac{ax^m}{bx^n}$

$f(x) = \dfrac{(x-1)(x+2)}{(x-1)(x+7)}$ hole at $x = 1$

$f(x) = \dfrac{x+2}{x+7}, x \neq 1$

$\dfrac{x^2}{x^2} = 1$

vertical asymptote at $x = -7$

horizontal asymptote at $y = 1$

as $x \to \infty, y \to 1$
as $x \to -\infty, y \to 1$

http://changestartsintheheart.wordpress.com/tag/asymptote/
http://www.theoildrum.com/node/5110

Example 1. Graph the rational function $R(x) = \frac{2x-6}{x+4}$ by completing the following:

- Factor and simplify $R(x)$. State the domain and any holes.
- State the long-run behavior and any horizontal asymptote.
- Find the vertical intercept.
- Find any zeros and find any vertical asymptotes. State the behavior of the function around the zeros and vertical asymptotes (preferably by making a table).

$R(x) = \frac{2(x-3)}{x+4}$ Domain: $\{x \mid x \neq -4\}$, no holes

Horizontal Asymptote: $\frac{2x}{x} = 2$ $y=2$ as $x \to \infty, y \to 2$ as $x \to -\infty, y \to 2$

$R(0) = \frac{2(0-3)}{0+4}$

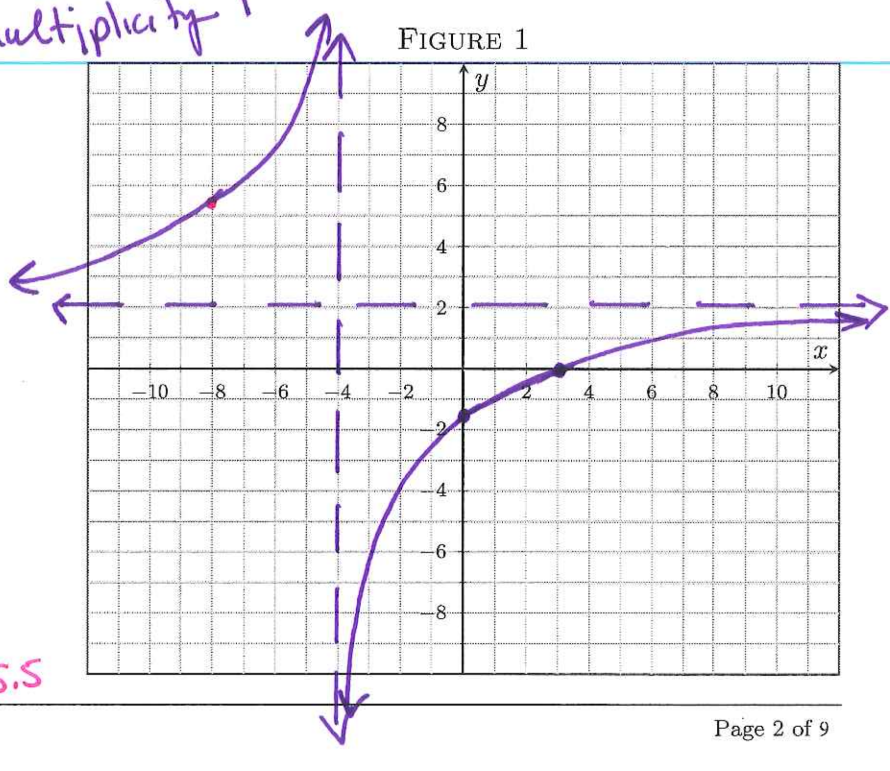
$= \frac{\cancel{2}^{1}(-3)}{\cancel{4}_{2}}$

$= -\frac{3}{2}$ $(0, -\frac{3}{2})$

Zeros: $x-3=0$

$x=3$, multiplicity 1

FIGURE 1



Vertical Asymptote: $x=-4$, multiplicity 1

Plot 1 point $x=-8$

$R(-8) = \frac{2(-8)-6}{-8+4}$

$= \frac{-16-6}{-4}$

$= \frac{-22}{-4} = +\frac{11}{2} = 5.5$

Example 2. Graph the rational function $R(x) = \frac{8}{x^2 - 4}$ by completing the following:

- Factor and simplify $R(x)$. State the domain and any holes.
- State the long-run behavior and any horizontal asymptote.
- Find the vertical intercept.
- Find any zeros and find any vertical asymptotes. State the behavior of the function around the zeros and vertical asymptotes (preferably by making a table).

$R(x) = \frac{8}{(x+2)(x-2)}$ Domain: $\{x \mid x \neq 2, -2\}$, no holes

Horizontal Asymptote: $\frac{8}{x^2} \to 0$ $y = 0$ as $x \to \infty, y \to 0$; as $x \to -\infty, y \to 0$

$R(0) = \frac{8}{0^2 - 4} = \frac{8}{-4} = -2$

Zeros: none

Vertical Asymptotes: $x = -2, 2$, mult 1

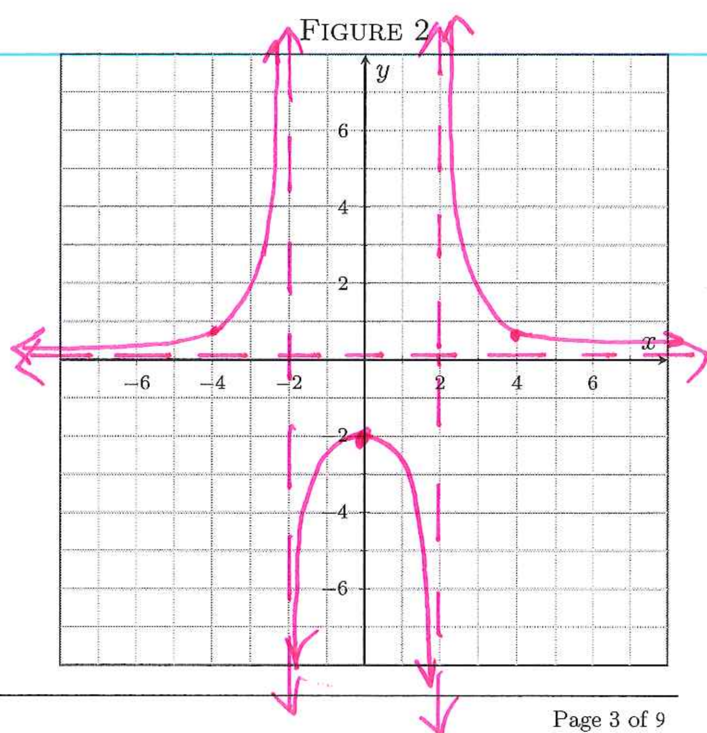
Checkpoints:

$x = 4$

$R(4) = \frac{8}{4^2 - 4} = \frac{8}{12} = \frac{2}{3}$

$R(-4) = \frac{8}{(-4)^2 - 4} = \frac{2}{3}$

FIGURE 2



Example 3. Graph the rational function $R(x) = \dfrac{x^2 - 4x + 3}{x - 3}$ by completing the following:

- Factor and simplify $R(x)$. State the domain and any holes.
- State the long-run behavior and any horizontal asymptote.
- Find the vertical intercept.
- Find any zeros and find any vertical asymptotes. State the behavior of the function around the zeros and vertical asymptotes (preferably by making a table).

$\{x \mid x \neq 3\}$

$R(x) = \dfrac{\cancel{(x-3)}(x-1)}{\cancel{x-3}} = R(x) = x - 1,\ x \neq 3$ hole at 3

$y = x - 1$

as $x \to \infty$, $y \to \infty$

as $x \to \infty$, $y \to -\infty$

no horizontal asymptotes

no vertical asymptotes

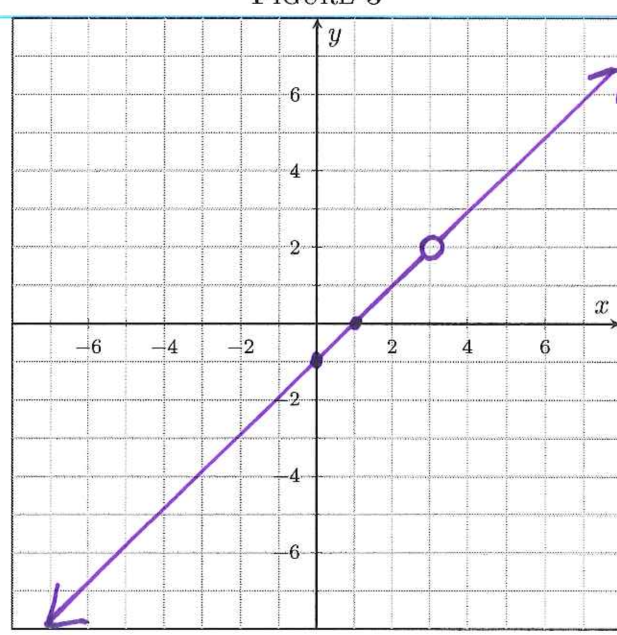
zero: 1

y-int: -1

could also write as a piecewise function

$$R(x) = \begin{cases} x - 1, & x < 3 \\ x - 1, & x > 3 \end{cases}$$

FIGURE 3



Example 4. Graph the rational function $R(x) = \dfrac{3x-6}{x^2+x-6}$ by completing the following:

- Factor and simplify $R(x)$. State the domain and any holes.
- State the long-run behavior and any horizontal asymptote.
- Find the vertical intercept.
- Find any zeros and find any vertical asymptotes. State the behavior of the function around the zeros and vertical asymptotes (preferably by making a table).

$R(x) = \dfrac{3\cancel{(x-2)}}{(x+3)\cancel{(x-2)}}$, hole at $x=2$ $(2, \frac{3}{5})$

$R(2) = \dfrac{3}{2+3} = \dfrac{3}{5}$

$R(x) = \dfrac{3}{x+3}$, $x \neq 2$ Domain: $\{x \mid x \neq -3, 2\}$

H.A. $\dfrac{3}{x} \to 0$ $y=0$

as $x \to \infty$, $y \to 0$
as $x \to -\infty$, $y \to 0$

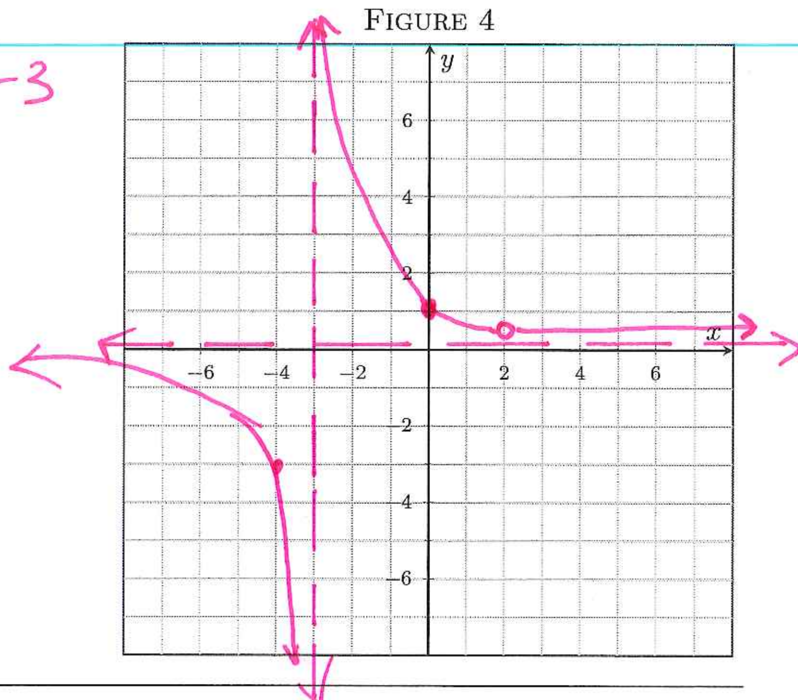
zeros: none

V.A: $x=-3$, mult 1

$R(0) = \dfrac{3}{0+3} = \dfrac{3}{3} = 1$

$R(-4) = \dfrac{3}{-4+3} = \dfrac{3}{-1} = -3$
$(-4,-3)$

FIGURE 4



How to find a possible formula for a rational function:

- State any zeros. Use these to determine factors and the multiplicity of each factor that appears in the numerator.
- State any vertical asymptotes. Use these to determine factors and the multiplicity of each factor that appears in the denominator.
- If a "hole" appears at $x = a$, then put the factor $(x - a)$ in both the numerator and denominator.
- Use one other point to determine if there is a constant factor other than 1.

Example 5. Find a possible formula for the rational function graphed in Figure 5.

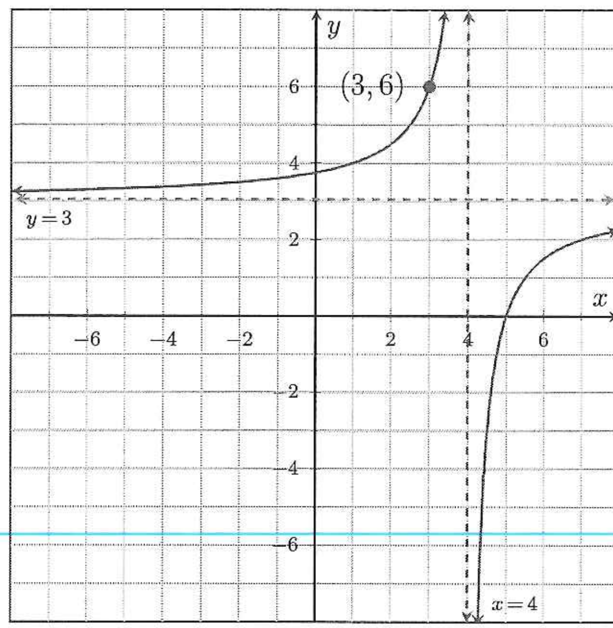
zeros: $x=5$, mult 1

$(x-5)$ numerator

V.A.: $x=4$, mult 1

$(x-4)$ denonom

FIGURE 5



$$y = \frac{k(x-5)}{(x-4)}$$

$$6 = \frac{k(3-5)}{(3-4)}$$

$$6 = \frac{k(-2)}{-1}$$

H.A. $= 3$

$$\frac{6}{2} = \frac{2k}{2}$$

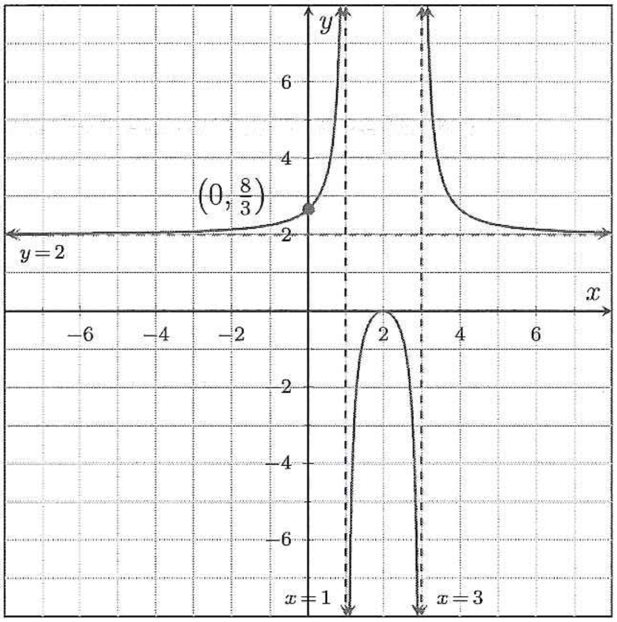
$$3 = k$$

$$f(x) = \frac{3(x-5)}{(x-4)}$$

Example 6. Find a possible formula for the rational function graphed in Figure 6.

$$y = \frac{k(x-2)^2}{(x-1)(x-3)}$$ ← zero ↖ V.A.

FIGURE 6



$$\frac{kx^2}{x^2} = k$$

$$\frac{8}{3} = \frac{k(0-2)^2}{(0-1)(0-3)}$$

$$\frac{8}{3} = \frac{k(4)}{3}$$

$$\frac{8}{4} = \frac{4k}{4}$$

$$2 = k$$

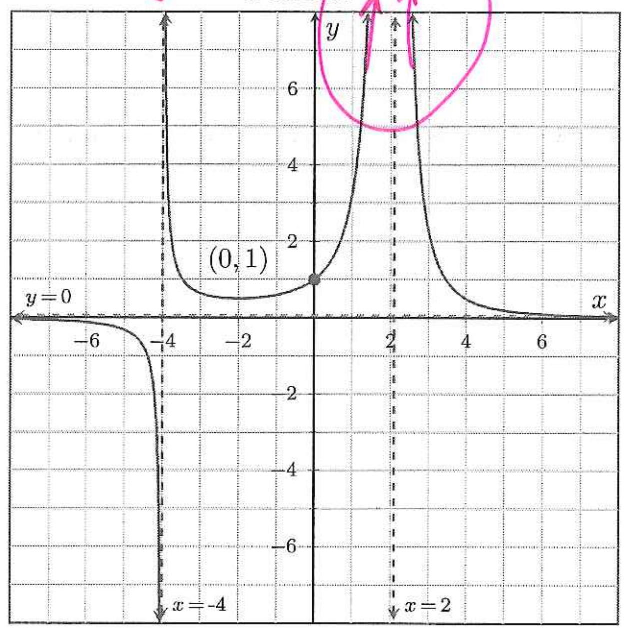
$$y = \frac{2(x-2)^2}{(x-1)(x-3)}$$

Example 7. Find a possible formula for the rational function graphed in Figure 7.

mult 1

multiplicity 2

FIGURE 7



$$y = \frac{k \cdot 1}{(x+4)(x-2)^2}$$

$$1 = \frac{k}{(0+4)(0-2)^2}$$

$$16\,\cancel{8} \cdot 1 = \frac{k}{\cancel{8}\,16} \cdot \cancel{8}\,16$$

$$\frac{16}{\cancel{8}} = k$$

$$y = \frac{\cancel{8}\,16}{(x+4)(x-2)}$$

$$y = \frac{16}{(x+4)(x-2)}$$

Group Work 1. Find a possible formula for the rational function graphed in Figure 8.

$$y = \frac{k(x-3)}{(x+1)}$$

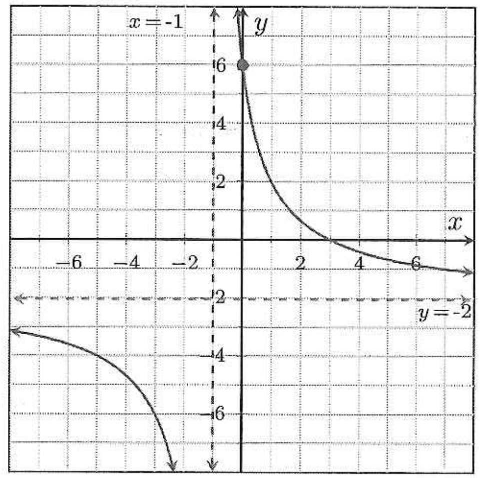
$k=-2$ because the degree of the top + bottom are the same

H.A. $= -2$

$$6 = \frac{k(0-3)}{(0+1)} \qquad 6 = -3k \qquad \frac{6}{-3} = \frac{-3k}{-3} \qquad -2 = k$$

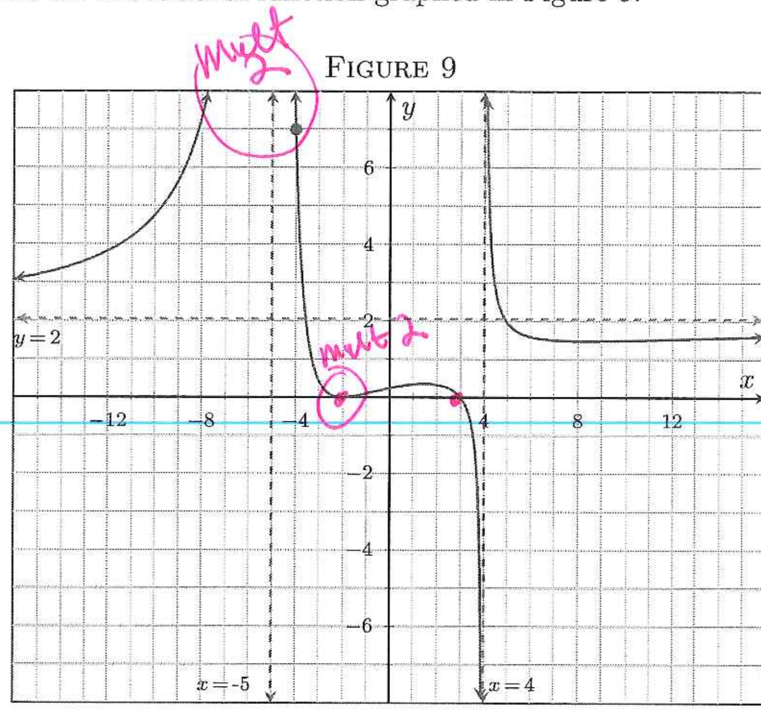
$$y = \frac{-2(x-3)}{(x+1)}$$

FIGURE 8



Group Work 2. Find a possible formula for the rational function graphed in Figure 9.

FIGURE 9



$$y = \frac{k(x-3)(x+2)^2}{(x-4)(x+5)^2}$$

$$\frac{kx^3}{1x^3} = k = 2$$

$(-4, 7)$

$$7 = \frac{k(-4-3)(-4+2)^2}{(-4-4)(-4+5)^2}$$

$$7 = \frac{k(-7)(4)}{(-8)}$$

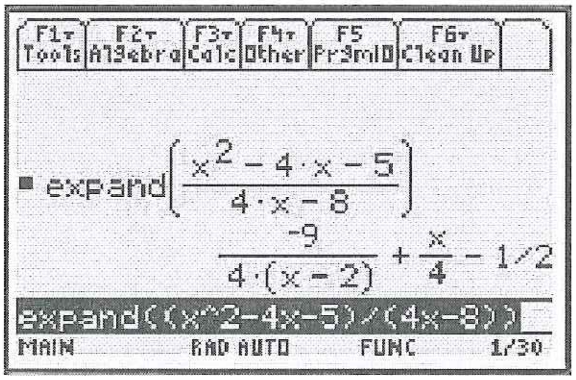
$$\frac{2}{7}\cdot 7 = k \cdot \frac{7}{2} \cdot \frac{2}{7}$$

$2 = k$

$$f(x) = \frac{2(x-3)(x+2)^2}{(x-4)(x+5)^2}$$

Example 8. Oblique Asymptotes

The graph of $R(x) = \frac{x^2-4x-5}{4x-8}$ "looks like" the function defined by $f(x) = \frac{1}{4}x$ in the long run. We know that this function has an oblique asymptote as the degree of the numerator is 1 greater than the degree of the denominator. To determine the equation of the oblique asymptote, either polynomial long division or a graphing calculator are needed. Using the "expand" key on a graphing calculator to perform polynomial long division, we find:



We use this to write:

$$R(x) = \frac{x^2-4x-5}{4x-8}$$

$$R(x) = \frac{-9}{4x-8} + \frac{1}{4}x - \frac{1}{2}$$

The first term in the expanded $R(x)$, which is $\frac{-9}{4x-8}$, is the *remainder*. The expression $\frac{1}{4}x - \frac{1}{2}$ is used to determine the equation of the oblique asymptote, which is $y = \frac{1}{4}x - \frac{1}{2}$. The function and its oblique asymptote are graphed in Figure 10 below.

FIGURE 10