Math 111, Mon, 5/2

Hand back tests to keep, progress reports

Finish 4.3 + Supplement

New material: 4.4 + Supplement

Bonus – self assessment (up to 5 points) > due next Monday

Mission 3 (Individual) Handed out

Project due Wed, 5/11

Desmos: To restrict the domain

$y = 3x \{0 < x < 10\}$

Checkpoint 5 on 4.3 and 4.4 (only switching from log to exp and exp to log)

Solve each equation. Look for similarities and differences in the processes.

1. $3x^2 + 4 = 52$

$\frac{3x^2}{3} = \frac{48}{3}$

$\sqrt{x^2} = \sqrt{16}$

$x = \pm 4$

$\{\pm 4\}$

2. $4 - (x-1)^3 = 0$

$-(x-1)^3 = -4$

$\sqrt[3]{(x-1)^3} = \sqrt[3]{4}$

$x - 1 = \sqrt[3]{4}$

$x = 1 + \sqrt[3]{4}$

exact solution

$\{1 + \sqrt[3]{4}\}$

3. $\left(\sqrt[4]{x+4}\right)^4 = 7^4$

$x + 4 = 7^4$

$x + 4 = 2401$

$x = 2397$

$\{2397\}$

Similarities:

exponents & roots

used inverse functions

Differences:

even root → ±

Inverse Functions:

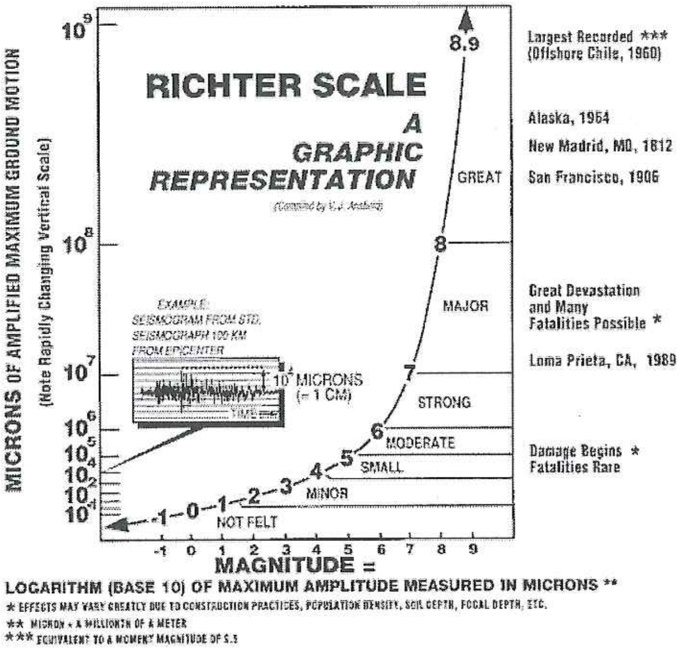
$y = x^2, x \geq 0$	$y = \sqrt{x}, x \geq 0$
Input: a number	Input: the square of a number
Output: the square of the number	Output: the number

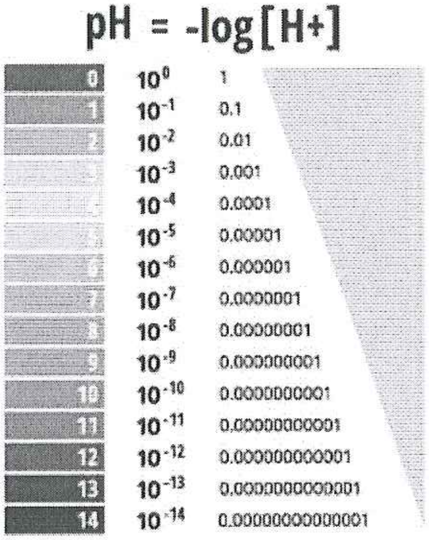
$y = \sqrt[3]{x}$	$y = x^3$
Input: the cube of a number	Input: a number
Output: the number	Output: the cube of the number

$y = 10^x$	$y = log_{10}x$
Input: an exponent	Input: a power of the given base (ex: 100)
Output: a power of the given base	Output: the exponent that gives that power

Logarithms are Exponents! Logarithms are Fun!

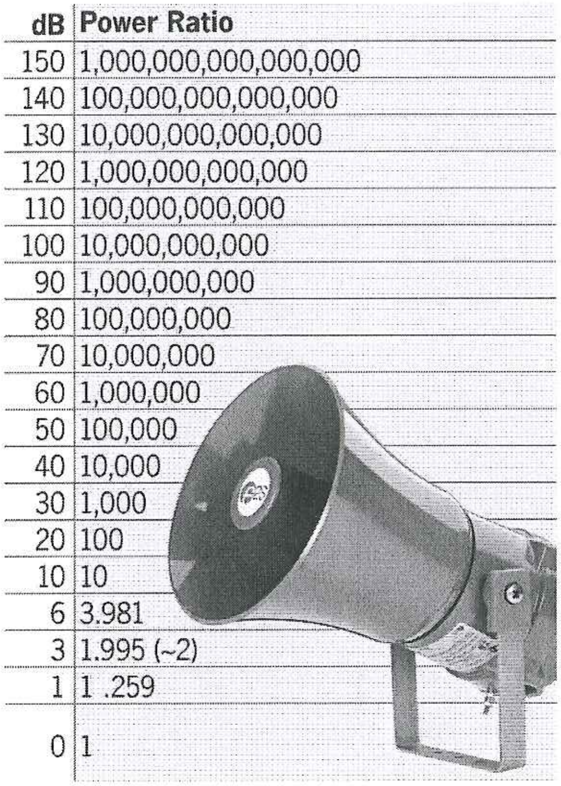
Logarithmic Scales: The Richter Scale, pH Levels and Decibels





$$pH = -\log[H+]$$

0	10^{0}	1
1	10^{-1}	0.1
2	10^{-2}	0.01
3	10^{-3}	0.001
4	10^{-4}	0.0001
5	10^{-5}	0.00001
6	10^{-6}	0.000001
7	10^{-7}	0.0000001
8	10^{-8}	0.00000001
9	10^{-9}	0.000000001
10	10^{-10}	0.0000000001
11	10^{-11}	0.00000000001
12	10^{-12}	0.000000000001
13	10^{-13}	0.0000000000001
14	10^{-14}	0.00000000000001



dB	Power Ratio
150	1,000,000,000,000,000
140	100,000,000,000,000
130	10,000,000,000,000
120	1,000,000,000,000
110	100,000,000,000
100	10,000,000,000
90	1,000,000,000
80	100,000,000
70	10,000,000
60	1,000,000
50	100,000
40	10,000
30	1,000
20	100
10	10
6	3.981
3	1.995 (~2)
1	1 .259
0	1

Math 111 Lecture Notes

SECTION 4.4: LOGARITHMIC FUNCTIONS

We looked at equations such as $2^{3x-1} = 32$ in the last section by rewriting 32 with a base of 2: $2^{3x-1} = 2^5$. But what if we have an equation such as $2^x = 10$? We know intuitively that $3 < x < 4$, but can we give an *exact* answer? We will need functions that are inverse functions to exponential functions in order to solve such equations.

The **logarithmic function to the base** a, where $a > 0$ and $a \neq 1$, is denoted by $y = \log_a(x)$ and is defined by

$$\text{exponent}\rightarrow y = \log_{\underset{\text{base}\nearrow}{a}}(\overset{\text{power}\swarrow}{x}) \quad \text{if and only if} \quad \overset{\text{power}\downarrow}{x} = \underset{\nwarrow\text{base}}{a}^{y}\leftarrow\text{exponent}$$

The **common logarithmic function** is the logarithmic function with base 10 given by $f(x) = \log(x)$. We write

$$\log(x) \quad \text{to represent} \quad \log_{10}(x)$$

The **natural logarithmic function** is the logarithmic function with base e given by $f(x) = \ln(x)$. We write

$\ln x$

$$\ln(x) \quad \text{to represent} \quad \log_e(x)$$

Example 1. Solve the equation $2^x = 10$ by converting from exponential form to logarithmic form.

$$\underset{\text{base}\nearrow}{2}^{x\leftarrow\text{exponent}} = \underset{\uparrow\ \text{power}}{10}$$

$x = \log_2 10$ exact solution

$x = \dfrac{\log 10}{\log 2}$ approximate solution

next section $\nearrow$

$x \approx 3.32$

$y = \log_a x$ iff $x = a^y$

Example 2. Change the exponential statement to an equivalent statement using logarithms.

(a) $3^t = 5$

$\log_3 5 = t$

(c) $10^m = 7$

$\log_{10} 7 = m$

$\log 7 = m$

(b) $\left(\frac{1}{4}\right)^x = 6$

$\log_{(1/4)} 6 = x$

equivalent to:

$$\boxed{\begin{array}{c} 4^{-x} = 6 \\ \log_4 6 = -x \end{array}}$$

(d) $e^{2t} = 5$

$\log_e 5 = 2t$

or

$\ln 5 = 2t$

Example 3. Change the logarithmic statement to an equivalent statement using exponents.

(a) $\log_3(5) = x$

$3^x = 5$

(c) $\log(x) = \frac{1}{2}$

$10^{\frac{1}{2}} = x$

(b) $\log_2(y + 1) = -3$

$2^{-3} = y + 1$

(d) $\ln(x) = 5$

$e^5 = x$

Example 4. If $f(x) = 2^x$, then the inverse function of f is given by $f^{-1}(x) = \log_2(x)$.

(a) Complete Table 1 and then sketch the graph of $y = f(x)$ in Figure 1.

(b) Use the properties of inverse functions to graph $y = f^{-1}(x)$.

(c) Use the properties of inverse functions to complete Table 2.

(d) Identify the domain, range and any asymptotes for each function.

FIGURE 1

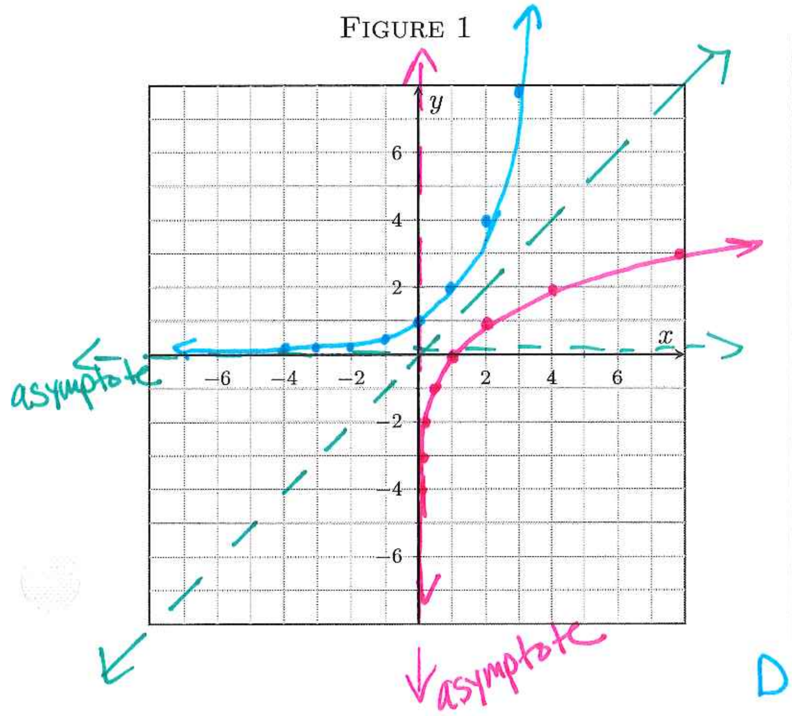


TABLE 1

x	2^x
-4	$\frac{1}{16}$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

D: $\mathbb{R}$

R: $(0,\infty)$

$\{y \mid y>0\}$

Asymptote

$y=0$

(x-axis)

TABLE 2

x	$\log_2(x)$
$\frac{1}{16}$	-4
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

D: $(0,\infty)$

R: $\mathbb{R}$

Asymptote:

$x=0$

(y-axis)

input to log must be >0

Example 5. State the domain of the following functions using interval notation.

$(-x+8) \; -(x-8)$

(a) $f(x) = \log_6(x-3)$

shift right 3

D: $(3,\infty)$

R: $\mathbb{R}$

asymptote: $x=3$

$x-3>0$

$x>3$

(b) $g(x) = \log(8-x)$

horizontal flip

right ~~left~~ 8

D: $(-\infty,8)$

$8-x>0$

$\frac{-x}{-1} > \frac{-8}{-1}$

$x<8$

(c) $h(x) = \log_5(x^2-4)$

set input >0

$x^2-4>0$

$x^2>4$

$x>2$ or $x<-2$

Example 6. Find the exact value of each logarithmic expression without using a calculator.

(a) $\log_5(25) = x$

$5^x = 25$

$x = 2$

$\log_5(25) = 2$

(b) $\log_3(3^{-2}) = -2$

$3^x = 3^{-2}$

$x = -2$

(c) $\ln(e^5) = 5$

$e^x = e^5$

$x = 5$

(d) $\log_2(32) = 5$

$2^x = 32$

$x = 5$

(e) $\log_2\left(\frac{1}{16}\right) = -4$

$2^x = \frac{1}{16}$

$x = -4$

(f) $\ln\left(\frac{1}{e}\right) = -1$

$e^x = \frac{1}{e}$

$x = -1$

(g) $\log(1000) = 3$

$10^x = 1000$

$x = 3$

(h) $\log_4(1) = 0$

$4^x = 1$

$x = 0$

(i) $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$

$6^x = \sqrt{6}$ $\Big\}$ $6^x = (6)^{1/2}$

$x = \frac{1}{2}$

Group Work 1. Find the exact value of each logarithmic expression.

(a) $\log_4(64) = 3$ (b) $\log_6\left(\frac{1}{36}\right) = -2$ (c) $\log(100) = 2$ (d) $\ln(e) = 1$

$y = \log_a x$ iff $x = a^y$

Example 7. Solve the following equations. Check all proposed solutions in your calculator and state the solution set.

(a) $e^{-2x+1} = 13$

$\log_e 13 = -2x + 1$

$\ln 13 = -2x + 1$

$\quad -1 \qquad -1$

$\frac{\ln(13) - 1}{-2} = \frac{-2x}{-2}$

$\frac{\ln(13)-1}{-2} = x$

$\frac{-\ln(13)+1}{2}$

$\frac{1-\ln(13)}{2}$

or

$\frac{1 - \ln 13}{2} = x$

$x \approx -.7825$

$\left\{\frac{1-\ln 13}{2}\right\}$

(c) $\log_3(5x - 4) = 2$

$3^2 = 5x - 4$

$9 = 5x - 4$

$\frac{13}{5} = \frac{5x}{5}$

$x = \frac{13}{5}$

$\left\{\frac{13}{5}\right\}$

(b) $\frac{8 \cdot 10^{5x}}{8} = \frac{3}{8}$

$10^{5x} = \frac{3}{8}$

$\frac{\log(3/8)}{5} = \frac{5x}{5}$

$\frac{1}{5}\log(3/8) = x$

$\left\{\frac{1}{5}\log(3/8)\right\}$

(d) $\log_2(x^2 + 1) = 3$

$2^3 = x^2 + 1$

$8 = x^2 + 1$

$7 = x^2$

$\pm\sqrt{7} = x$

$\{\pm\sqrt{7}\}$

Example 8. Let $p(h)$ be the atmospheric pressure on an object (measured in millimeters of mercury) that is h kilometers above sea level. The function p can be modeled by

$$\underbrace{p(h)}_{620} = 760e^{-0.145h}$$

Find the height of a mountain where the atmospheric pressure is 620 millimeters of mercury.

$$\frac{620}{760} = \frac{760e^{-.145h}}{760}$$

$$.815789 = e^{-.145h}$$

$$\frac{\ln(.815789)}{-.145} = \frac{-.145h}{-.145}$$

$$1.4_{km} \approx h$$

The height of
the mountain
is 1.4 km

Example 9. The pH of a chemical solution is given by the formula

$$\mathrm{pH} = -\log_{10}\left[\mathrm{H}^+\right]$$

where $\left[\mathrm{H}^+\right]$ is the concentration of hydrogen ions in moles per liter. Values of pH range from 0 (acidic) to 14 (alkaline).

(a) What is the pH of a solution for which the concentration of hydrogen ions ($\left[\mathrm{H}^+\right]$) is 0.01?

$$pH = -\log_{10}(.01)$$
$$pH = 2$$

$$\left[\log_{10}(.01) = x\right]$$
$$10^x = .01$$
$$10^x = \frac{1}{100}$$
$$10^{-2} = \frac{1}{100}$$

(b) What is the concentration of hydrogen ions ($\left[\mathrm{H}^+\right]$) in a banana with a pH of 4.5?

$$\frac{4.5}{-1} = \frac{-\log_{10}([H^+])}{-1}$$

$$-4.5 = \log_{10}([H^+])$$

$$10^{-4.5} = [H^+]$$

$$.000032 = [H^+]$$

3. A population decreases at a rate of 13.2% per 5 years. Find the approximate value for the following:

- **a.** 1-year factor of decay and 1-year rate of decay.
- **b.** 5-year factor of decay and 5-year rate of decay.
- **c.** 10-year factor of decay and 10-year rate of decay.

SUPPLEMENTAL PROBLEMS FOR §4.4

EXAMPLE: The graph of $f(x) = \log_a(x)$ is given in Figure 17. Find a. (Note that the points $(1, 0)$ and $(9, 2)$ are on the graph of f.)

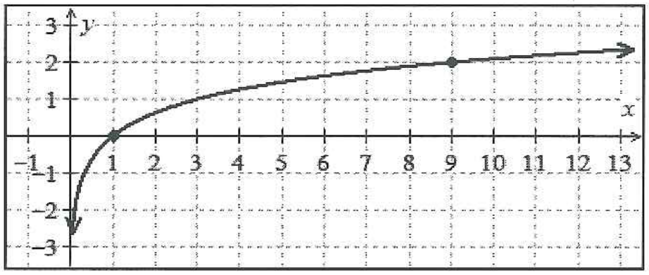


Figure 17: $f(x) = \log_a(x)$

$a^0 = 1$
$\log_a x = y$
$\log_a 1 = y$
$a^y = 1$
$a^0 = 1$ $(1,0)$

Solution:

Since the function has form $f(x) = \log_a(x)$ and since the point $(9, 2)$ is on the graph, we know that $f(9) = 2$. Thus,

$(9,2)$
$\log_a x = y$
$\log_a 9 = 2$

$$f(9) = 2$$

$\Rightarrow \log_a(9) = 2$ (since $f(9) = \log_a(9)$)

$\Rightarrow a^2 = 9$ (translate the logarithmic statement into an exponential one) $a^2 = 9$

$\Rightarrow a = 3$ (take the positive square root of 9 because bases of logs are positive) $3^2 = 9$

$a = 3$

Notice that we didn't attempt to use $(1, 0)$, the other obvious point on the graph of $f(x) = \log_a(x)$, to find a. Why not? (The point $(1, 0)$ is on the graph of *all* functions of the form $f(x) = \log_a(x)$ so it doesn't provide information that will help us find the particular function graphed here.)