For each function find and simplify $f(-x)$. What patterns do you notice?

a. $f(x) = 3x^4 - 6x^3 - 10x^2 + x - 3$

b. $f(x) = -2x^4 - 7x^2 - 3$

c. $f(x) = 5x^3 + 4x$

Math 111 Lecture Notes

Section 1.3: Properties of Functions

A function f is **even** if for every x in the domain of f it holds that $f(-x) = f(x)$. Visually, an even function is *symmetric about the y-axis*.

A function f is **odd** if for every x in the domain of f it holds that $f(-x) = -f(x)$. Visually, an odd function is *symmetric about the origin*.

Example 1. Two classic examples of even and odd functions are $f(x) = x^2$ and $g(x) = x^3$, respectively, as shown in Figures 1 and 2 below.

Figure 1. Graph of $y = f(x)$

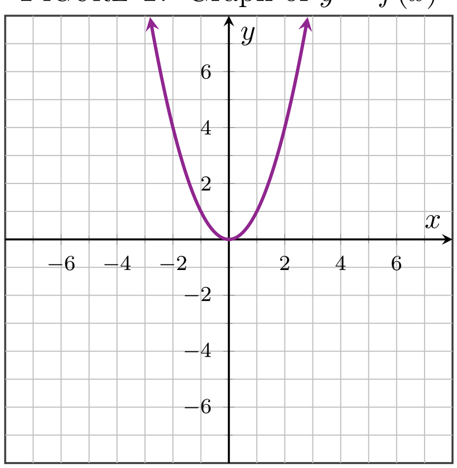
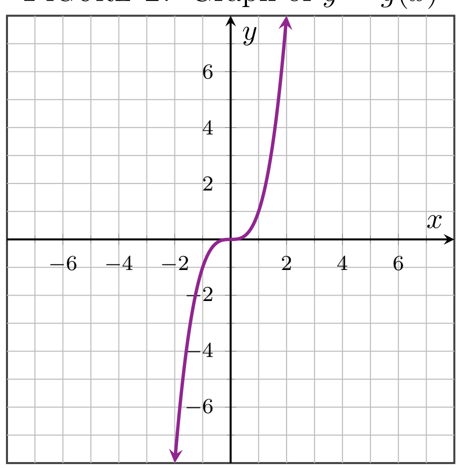


Figure 2. Graph of $y = g(x)$



Algebraically verify that f is an even function and that g is an odd function.

Example 2. Algebraically determine if the following functions are even, odd or neither.

(a) $h(x) = x^3 - x$

(c) $f(t) = t^3 + 1$

(b) $g(t) = \frac{1}{2}t^4 - 1$

(d) $f(x) = |x| - 4$

Figure 3. $y = h(x)$

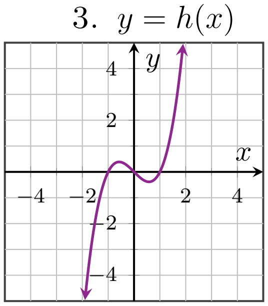


Figure 4. $y = g(t)$

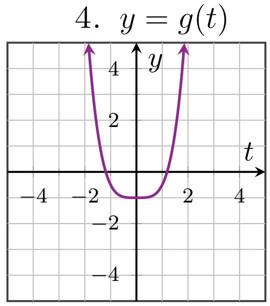


Figure 5. $y = f(t)$

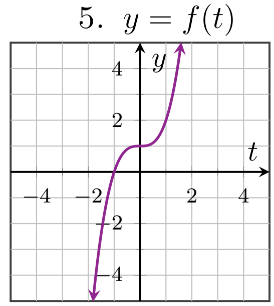
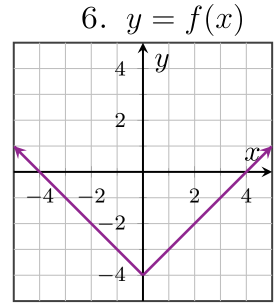


Figure 6. $y = f(x)$



Example 3. Algebraically determine if the function f defined by $f(x) = -\dfrac{2x^3 - x}{3x^4 + 5x^2}$ is even, odd or neither.

Group Work 1. Determine if the following functions are even, odd or neither.

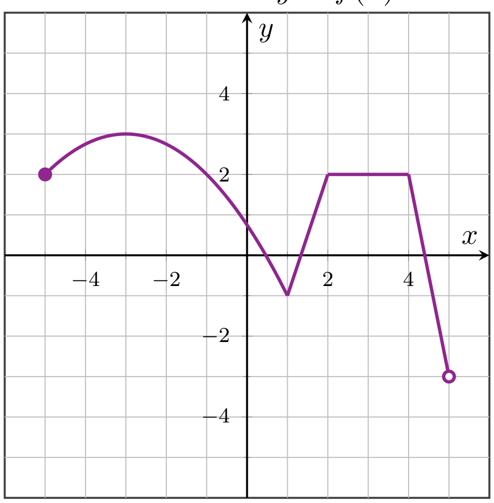
(a) $g(x) = \dfrac{x^2}{x^4 + 5}$

(b) $f(x) = 5x^3 + 3x^2$

A function f is **increasing** on an open interval I if for every x_1 and x_2 in I with $x_1 < x_2$ we have $f(x_2) > f(x_1)$.
A function f is **decreasing** on an open interval I if for every x_1 and x_2 in I with $x_1 < x_2$ we have $f(x_2) < f(x_1)$.
A function f is **constant** on an open interval I if for every x_1 and x_2 in I with $x_1 < x_2$ we have $f(x_2) = f(x_1)$.

Example 4. Determine the following for the function f graphed in Figure 7. State each using interval notation.

FIGURE 7. $y = f(x)$



(a) Increasing:

(b) Decreasing:

(c) Constant:

(d) Domain of f:

(e) Range of f:

A function has a **local maximum** at c if there exists an open interval I containing c so that for all x not equal to c in I, it holds that $f(x) < f(c)$. The output $f(c)$ is referred to as the **local maximum** of f.
A function has a **local minimum** at c if there exists an open interval I containing c so that for all x not equal to c in I, it holds that $f(x) > f(c)$. The output $f(c)$ is referred to as the **local minimum** of f.

Example 5. Use Figure 7 to answer the following:

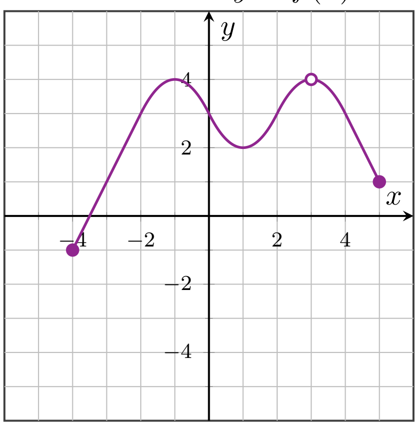
(a) Identify all local maximum values of f and state where they occur.

(b) Identify all local minimum values of f and state where they occur.

Let f be a function defined on an interval I.
A function has an **absolute maximum** at u if it holds that $f(x) \leq f(u)$ for all x in the interval I. The output $f(u)$ is referred to as the **absolute maximum** of f.
A function has an **absolute minimum** at u if it holds that $f(x) \geq f(u)$ for all x in the interval I. The output $f(u)$ is referred to as the **absolute minimum** of f.

Example 6. Use Figure 8 to answer the following:

FIGURE 8. $y = f(x)$



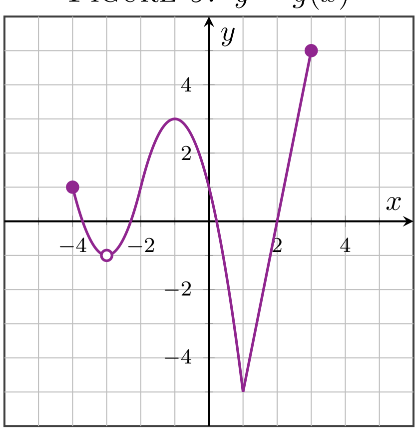
(a) Identify all absolute maximum values of f and state where they occur.

(b) Identify all absolute minimum values of f and state where they occur.

Group Work 2. Use Figure 9 to answer the following:

(a) Identify all local maximum values of g and state where they occur.

FIGURE 9. $y = g(x)$



(b) Identify all local minimum values of g and state where they occur.

(c) Identify all absolute maximum values of g and state where they occur.

(d) Identify all absolute minimum values of g and state where they occur.

Concavity

So far, we have looked at where a function is increasing and decreasing and where it attains maximum and minimum values. We will now study the concept of *concavity*. This concept involves looking at the rate at which a function increases or decreases.

The graph of a function f whose rate of change increases (becomes less negative or more positive as you move left to right) over an interval is **concave up** on that interval. Visually, the graph "bends upward."

The graph of a function f whose rate of change decreases (becomes less positive or more negative as you move left to right) over an interval is **concave down** on that interval. Visually, the graph "bends downward."

Figure 10. Concave UP ☺

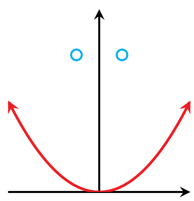
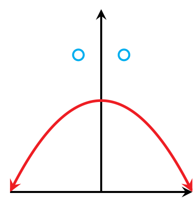


Figure 11. Concave DOWN ☹



Example 7. The function defined by $f(x) = x^2$ is concave up on its entire domain. Notice that it is **decreasing** on the interval $(-\infty, 0)$ and **increasing** on the interval $(0, \infty)$. The function defined by $f(x) = -x^2$ is concave down on its entire domain. Notice that it is **increasing** on the interval $(-\infty, 0)$ and **decreasing** on the interval $(0, \infty)$.

Figure 12. Graph of $y = x^2$

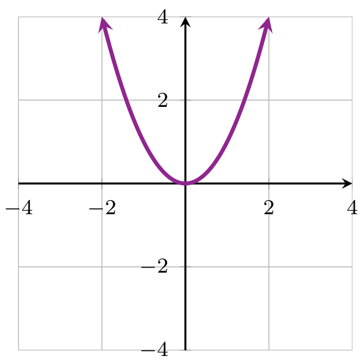
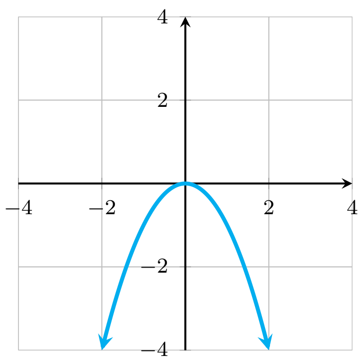
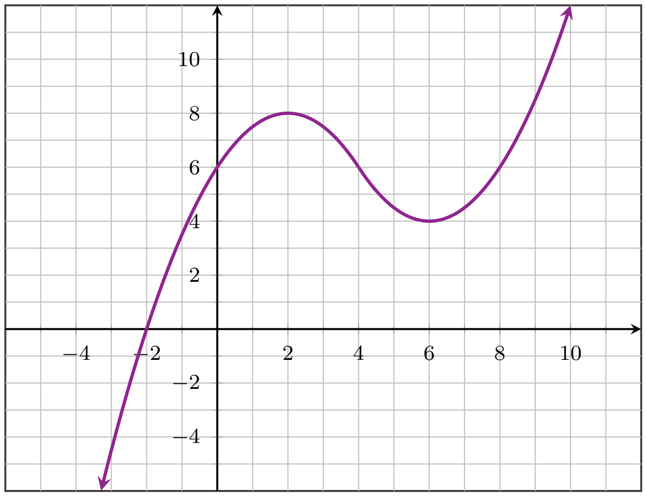


Figure 13. Graph of $y = -x^2$



Example 8. The graph of $y = h(x)$ is shown in Figure 14. Use this to answer the following.

FIGURE 14. Graph of $y = h(x)$



(a) State the interval(s) where h is positive.

(b) State the interval(s) where h is negative.

(c) State the interval(s) where h is increasing.

(d) State the interval(s) where h is decreasing.

(e) State the interval(s) where h is concave up.

(f) State the interval(s) where h is concave down.

(g) State any absolute maximum or absolute minimum values for h and where they occur.

(h) State any local maximum or local minimum values for h and where they occur.

Example 9. Graph the function defined by $k(x) = 2x^4 - 6x^3 - 6x^2 + 22x + 2$ on your calculator.

(a) Determine an appropriate window that shows the important features (such as the x-intercept(s), y-intercept, and any local maxima or minima).

(b) Use the MAXIMUM and MINIMUM features to find any local maxima and minima and where they occur.

(c) (Review) Use the ZERO feature and the VALUE feature to determine the x-intercepts and y-intercept.