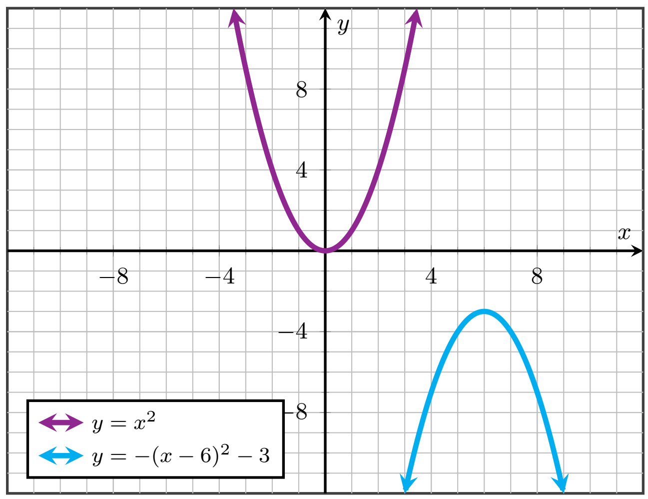
Math 111 Lecture Notes

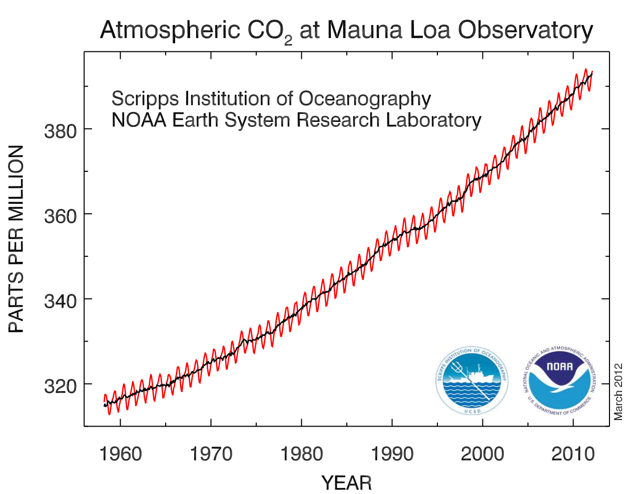
Section 1.5: Function Transformations

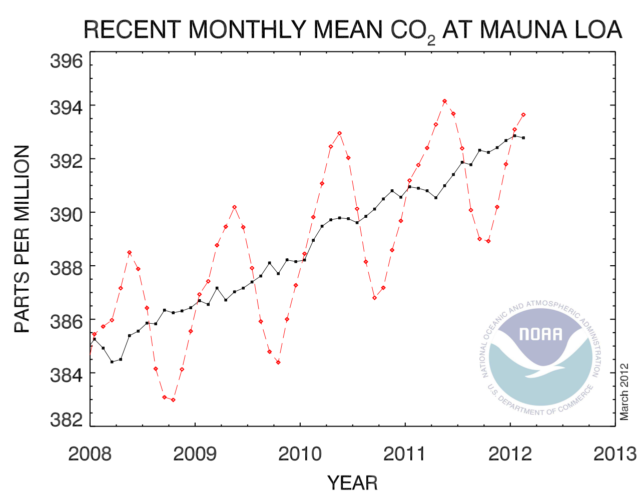
In this section, we will explore *function transformations*. We will explore these numerically (in tabular form), algebraically (as formulas), and graphically. When you studied the vertex form of a parabola, you were actually studying function transformations for a specific function—namely, $f(x) = x^2$. For example, when graphing $y = -(x-6)^2 - 3$, you know that the graph points downward and that the vertex is $(6, -3)$.

Figure 1



We could also say that the graph is reflected about the x-axis, shifted right 6 units, and then shifted down 3 units. In this course, we will be able to apply similar transformations to any function—not just parabolas! One such example is shown below. ☺





http://www.esrl.noaa.gov/gmd/ccgg/trends/

Let $y = f(x)$, where x is the number of months after January 1, 2011 and $f(x)$ is the amount of CO_2 in the atmosphere after x months. We will measure $f(x)$ in parts per million above 380 and restrict x to $-3 \leq x \leq 9$. The data for September 2010 through September 2011 is shown in Figure 2.

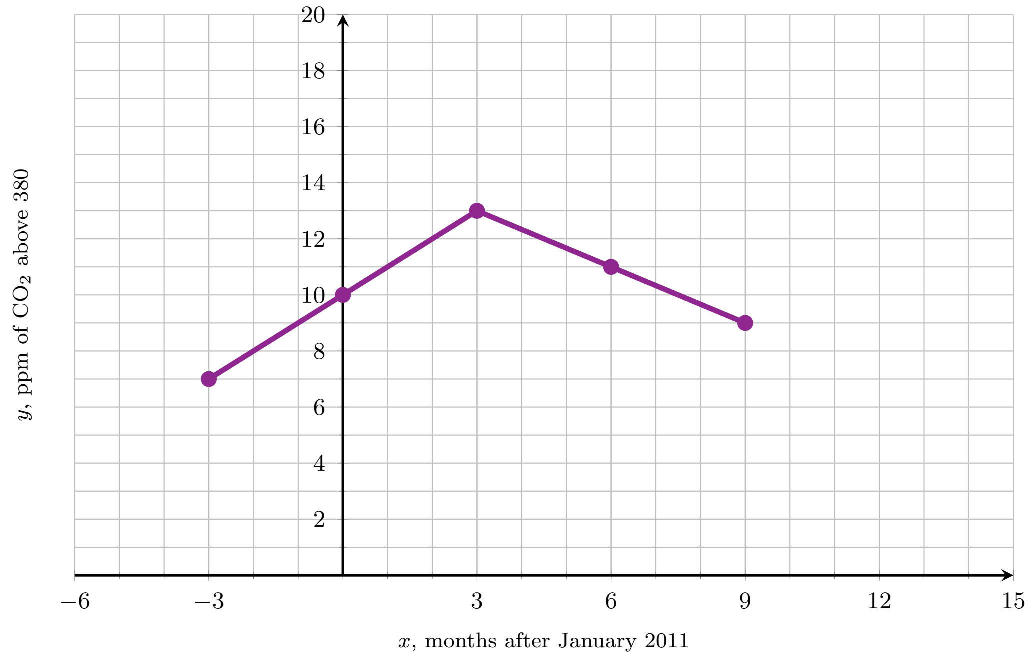
Vertical Shifts

Example 1. Complete Table 1 using the function values for f. What happens to the graph in each case? Sketch and label the graph of $y = f(x) + 4$ and the graph of $y = f(x) - 2$ in Figure 2.

Table 1

x	-3	0	3	6	9
$f(x)$	7	10	13	11	9
$f(x)+4$					
$f(x)-2$					

Figure 2



Summary of Vertical Shifts

The graph of $y = f(x) + k$ is transformation of the graph of $y = f(x)$.

- If $k > 0$, then the graph of the original function shifts ____________ by k units.
- If $k < 0$, then the graph of the original function shifts ____________ by k units.

Horizontal Shifts

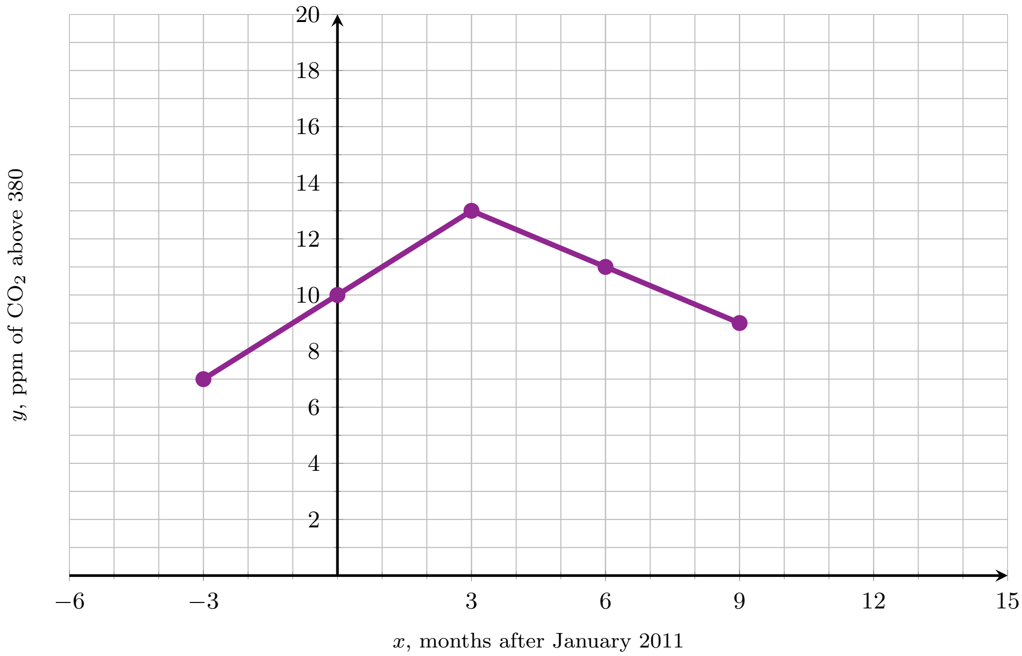
Horizontal shifts are not quite as straightforward as vertical shifts. The primary reason is that in order to shift the graph horizontally, we need to add or subtract from x—*before* we evaluate the function. The end result is that horizontal transformations work a bit backwards from what you may expect, as we will discover in the example below.

Example 2. Complete Table 2 using the function values for f. What happens to the graph in each case? Sketch and label the graph of $y = f(x+3)$ and the graph of $y = f(x-6)$ in Figure 3.

Table 2

x	-6	-3	0	3	6	9	12	15
$f(x)$	und.	7	10	13	11	9	und.	und.
$f(x+3)$								
$f(x-6)$								

Figure 3



Summary of Horizontal Shifts

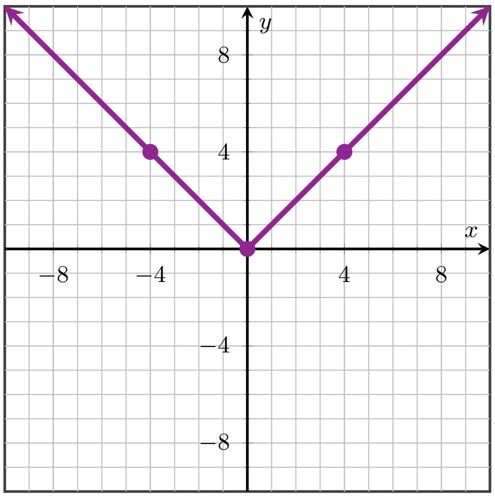
The graph of $y = f(x+h)$ is transformation of the graph of $y = f(x)$.

- If $h > 0$, then the graph of the original function shifts ____________ by h units.
- If $h < 0$, then the graph of the original function shifts ____________ by h units.

Example 3. For each function below, the "original" or "basic" function is $y = |x|$. Use the properties of horizontal and vertical shifts to graph the stated transformations. The full graph and 3 key points are given in each.

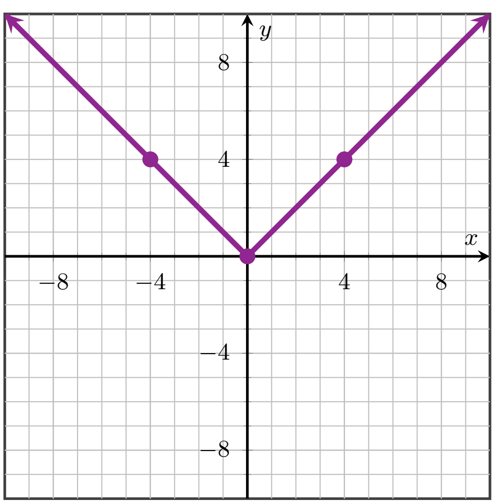
(a) Graph $y = |x| - 5$.

FIGURE 4



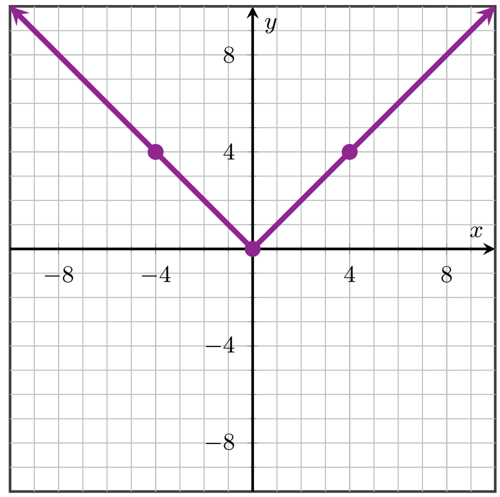
(b) Graph $y = |x + 4|$.

FIGURE 5



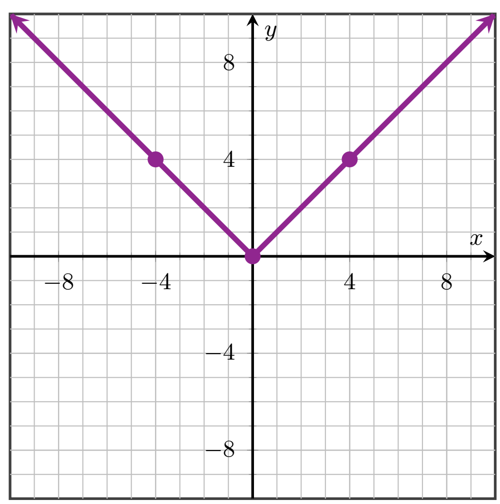
(c) Graph $y = |x + 2| - 1$.

FIGURE 6



(d) Graph $y = |x - 3| - 6$.

FIGURE 7



Vertical Stretches and Compressions

Example 4. Assume the base temperature setting for the thermostat in a house is 64°F. Let $g(x)$ be the number of degrees above 64°F x hours after 6am. Complete Table 3 using the function values for g. What happens to the graph in each case? Sketch and label the graph of $y = 2g(x)$ in Figure 8 and the graph of $y = \frac{1}{2}g(x)$ in Figure 9.

Table 3

x	-2	0	4	7	8
$g(x)$	-2	6	6	0	-2
$2g(x)$					
$\frac{1}{2}g(x)$					

Figure 8

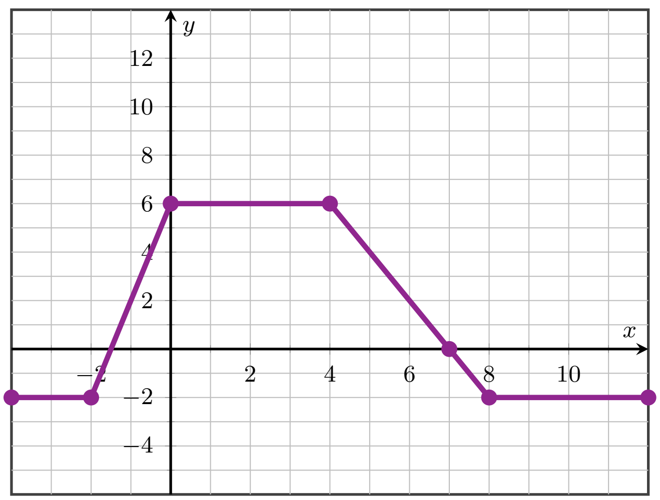
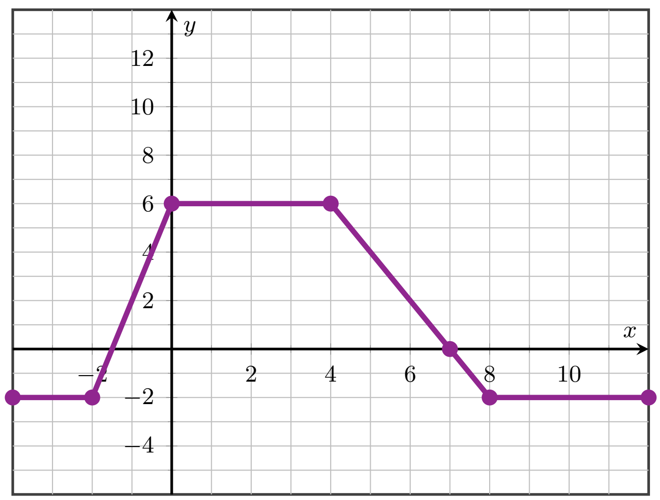


Figure 9



Summary of Vertical Stretches and Compressions

The graph of $y = Af(x)$ is transformation of the graph of $y = f(x)$. If

- If $|A| > 1$, then the graph of the original function ____________________ vertically by a factor of $|A|$.
- If $0 < |A| < 1$, then the graph of the original function ____________________ vertically by a factor of $|A|$.

Horizontal Stretches and Compressions

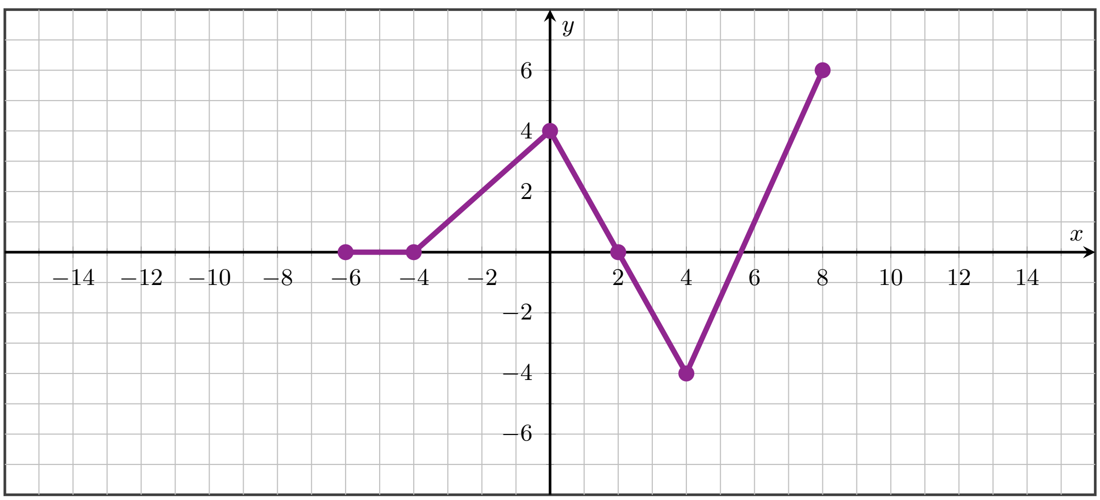
Horizontal stretches and compressions, much like horizontal shifts, work in a somewhat counterintuitive way. This again is a result of the fact that we will multiply x by a number *before* we evaluate the function.

Example 5. The graph of $y = h(x)$ is shown below. Complete Table 4 and then graph $y = h\left(\frac{1}{2}x\right)$ in Figure 10.

Table 4

x	-12	-8	-6	-4	0	2	4	8	16
$h(x)$	und.	und.	0	0	4	0	-4	6	und.
$h\left(\frac{1}{2}x\right)$									

Figure 10

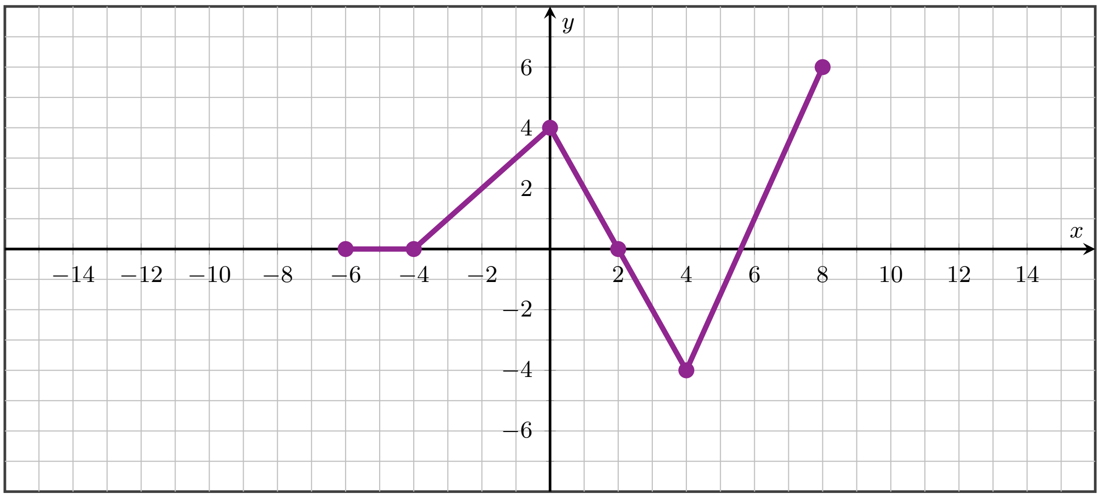


Example 6. The graph of $y = h(x)$ is shown below. Complete Table 5 and then graph $y = h(4x)$ in Figure 11. An "X" is placed where the function is defined but difficult to evaluate.

Table 5

x	-6	-4	-1.5	-1	0	0.5	1	2	4	8
$h(x)$	0	0	X	3	4	X	2	0	-4	6
$h(4x)$										

Figure 11



Summary of Horizontal Stretches and Compressions

The graph of $y = f(Bx)$ is transformation of the graph of $y = f(x)$.

- If $|B| > 1$, then the graph of the original function ____________ horizontally by a factor of $\frac{1}{|B|}$.
- If $0 < |B| < 1$, then the graph of the original function ____________ horizontally by a factor of $\frac{1}{|B|}$.

Horizontal and Vertical Reflections

Example 7. The graph of $y = h(x)$ is shown below. Complete Table 6 and then graph $y = -h(x)$ in Figure 12 and graph $y = h(-x)$ in Figure 13.

Table 6

x	-8	-6	-4	-2	0	2	4	8
$h(x)$	und.	0	0	2	4	0	-4	6
$-h(x)$								
$h(-x)$								

Figure 12

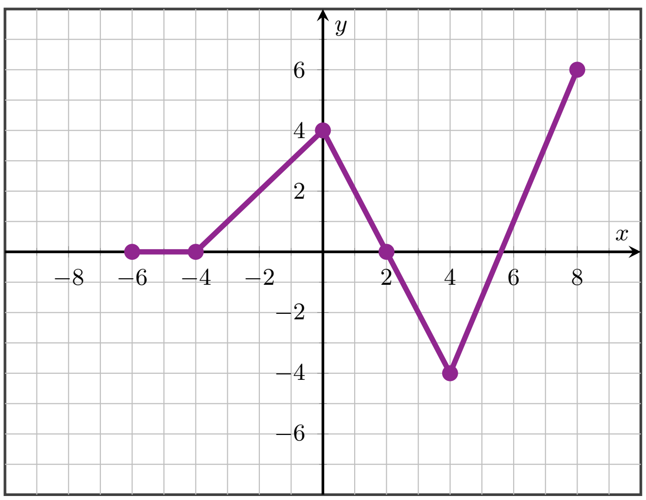
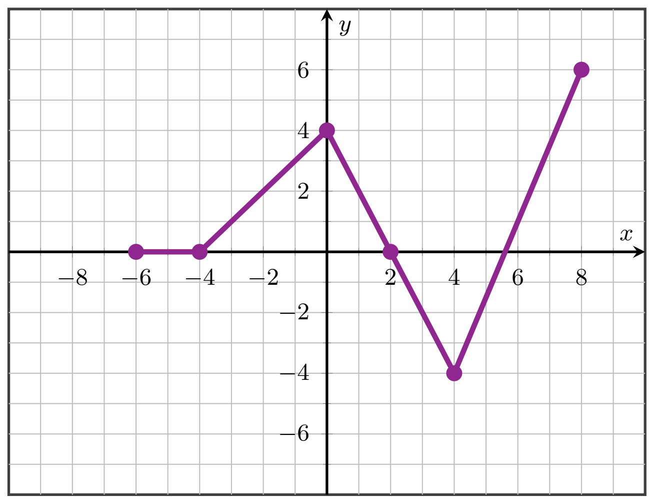


Figure 13



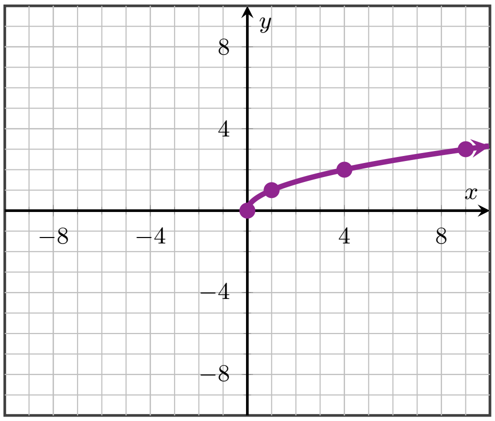
Summary of Horizontal and Vertical Reflections

- The graph of $y = -f(x)$ is transformation of the graph of $y = f(x)$. It reflects the graph of the original function across the ________________ axis.
- The graph of $y = f(-x)$ is transformation of the graph of $y = f(x)$. It reflects the graph of the original function across the ________________ axis.

Example 8. For each function below, the "original" or "basic" function is $y = \sqrt{x}$. Use the properties of horizontal and vertical stretches and compressions to graph the stated transformations. The full graph and 4 key points are given in each.

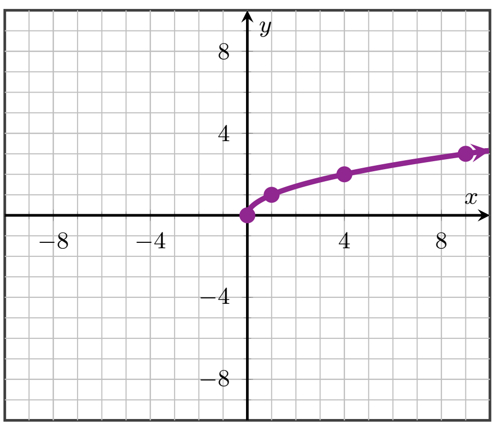
(a) Graph $y = 4\sqrt{x}$.

FIGURE 14



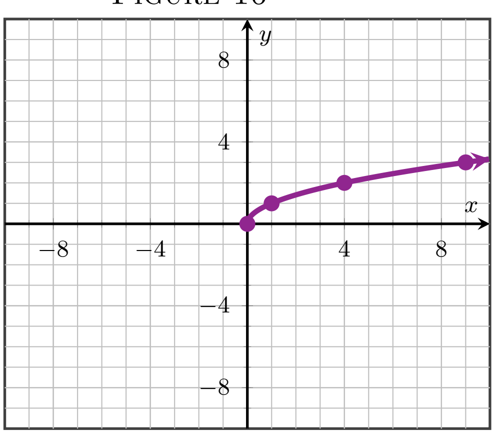
(b) Graph $y = \sqrt{\frac{1}{3}x}$.

FIGURE 15



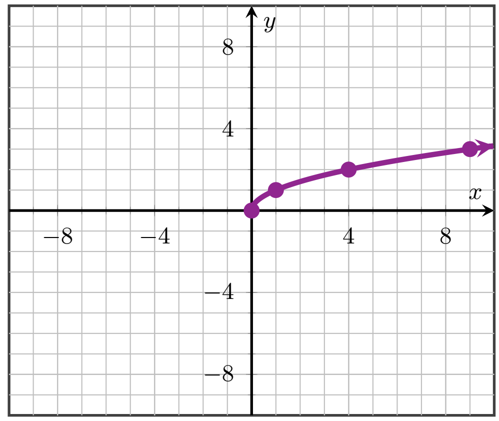
(c) Graph $y = -\sqrt{x}$.

FIGURE 16



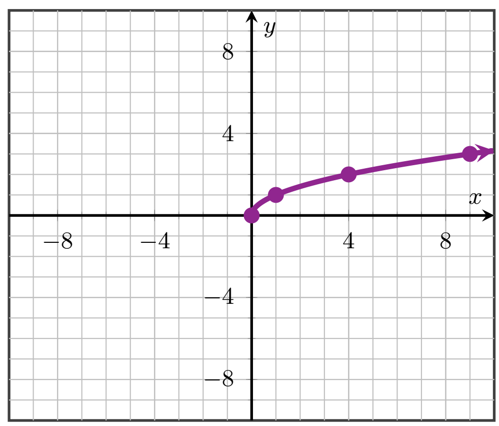
(d) Graph $y = \frac{1}{2}\sqrt{x}$.

FIGURE 17



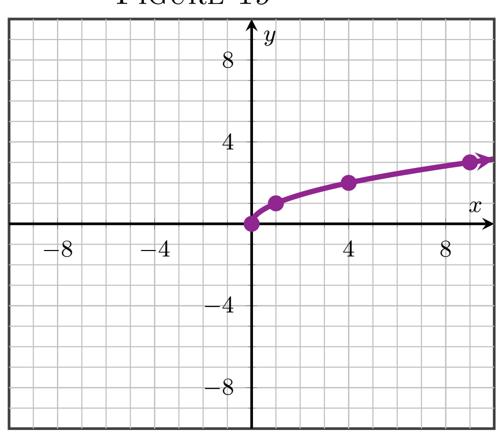
(e) Graph $y = \sqrt{2x}$.

FIGURE 18



(f) Graph $y = \sqrt{-x}$.

FIGURE 19



Example 9. The point $(4, 12)$ is on the graph of $y = f(x)$. Determine the point on the graph of...

(a) $y = f(x + 2) - 1$

(b) $y = 5f(x)$

(c) $y = -f(x - 5) + 4$

(d) $y = f\left(\frac{1}{3}x\right)$

(e) $y = f(-x) - 5$

(f) $y = 2f(4(x + 1)) - 3$

Example 10. For the function below, identify the original (or "basic") function and explain how the graph is a transformation of the graph of the original function. State all steps to this transformation in an appropriate order.

(a) $g(x) = 8\sqrt[3]{-4x}$

(b) $h(x) = -|2x + 6|$

(c) $j(x) = \frac{2}{3}\left(5(x-1)\right)^3 + 4$

Example 11. Let $g(x) = -(x-6)^2 - 3$.

(a) Identify the original (or "basic") function and explain how the graph of $y = g(x)$ is a transformation of the original function. State all steps to this transformation in an appropriate order.

(b) Compare the graph of $y = g(x)$ to the graph of $y = x^2$ after it has been shifted right 6 units, shifted down 3 units and THEN reflected about the x-axis.

FIGURE 20

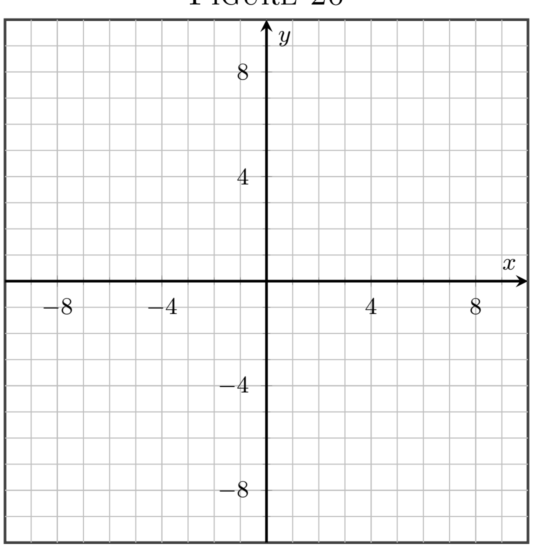
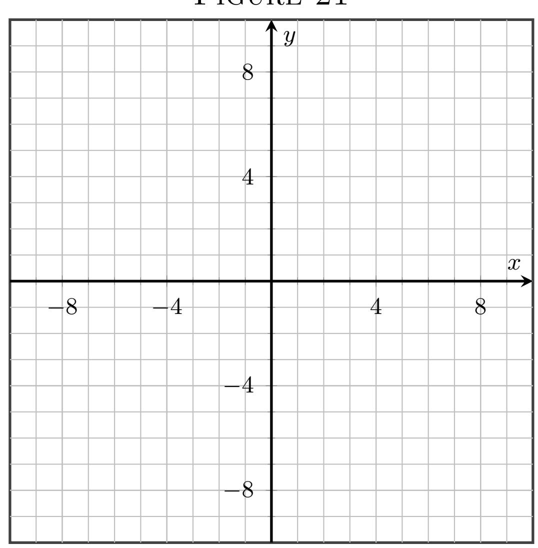
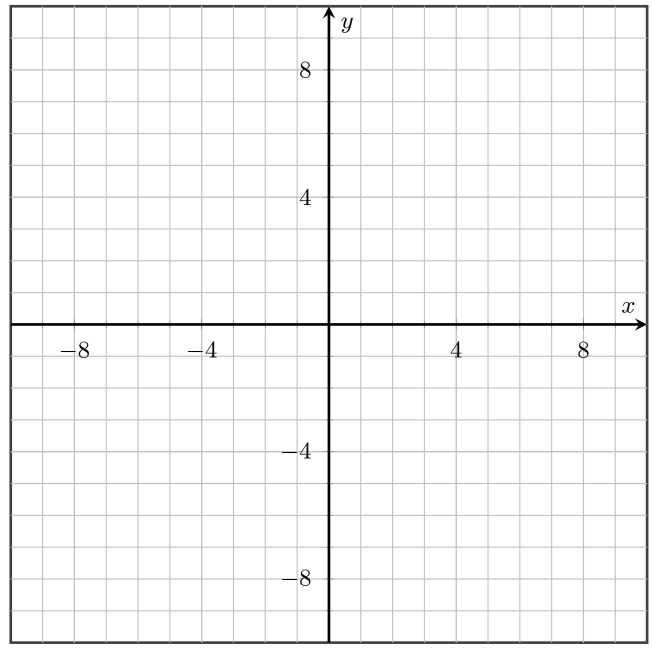


FIGURE 21



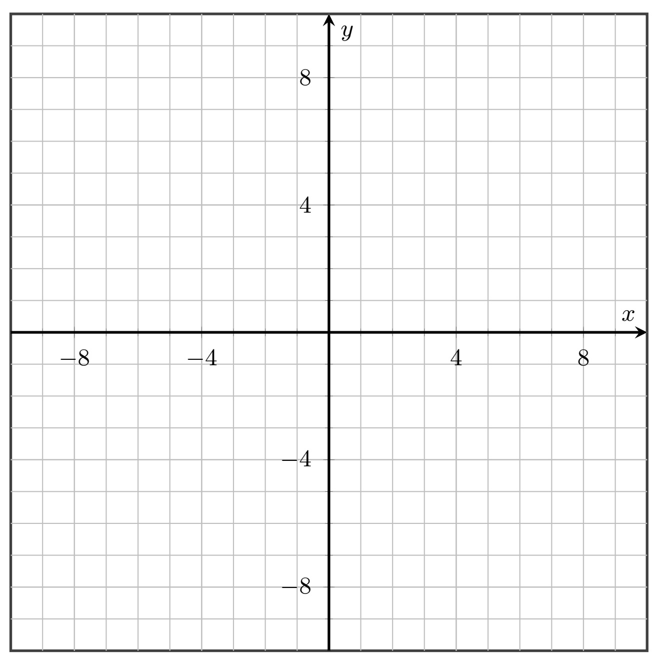
Example 12. Let $g(x) = \frac{1}{2}(x+5)^3 + 4$. Identify the original function and explain how the graph of $y = g(x)$ is a transformation of the graph of the original function. Then sketch a graph of $y = g(x)$ in Figure 22.

FIGURE 22



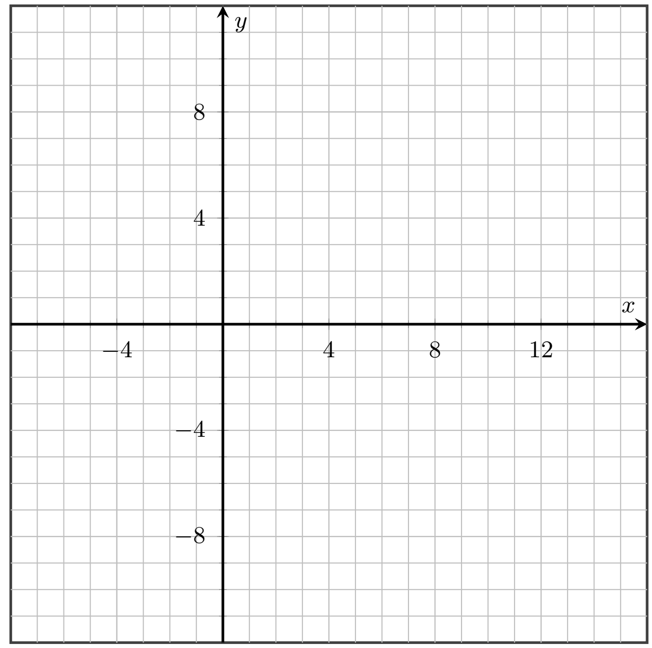
Example 13. Let $g(x) = \left|\frac{1}{2}x - 3\right| - 1$. Identify the original function and explain how the graph of $y = g(x)$ is a transformation of the graph of the original function. Then sketch a graph of $y = g(x)$ in Figure 23.

FIGURE 23



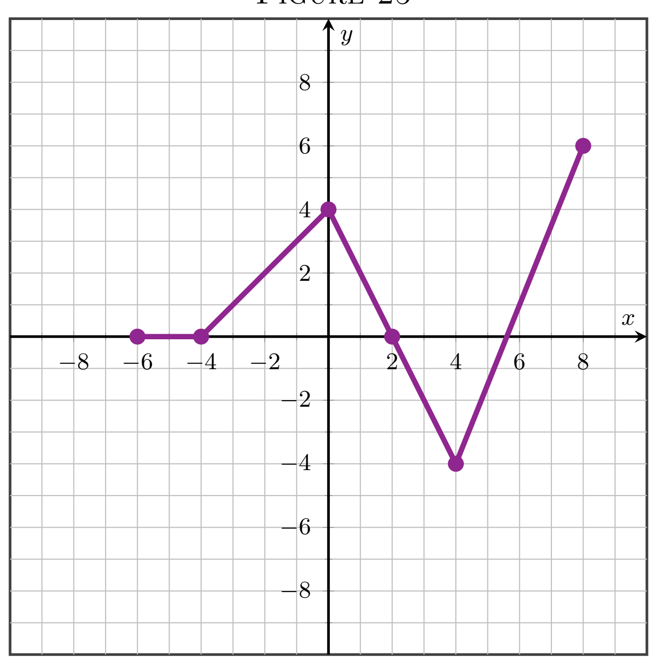
Example 14. Let $g(x) = \sqrt{-(x+3)} + 2$. Identify the original function and explain how the graph of $y = g(x)$ is a transformation of the graph of the original function. Then sketch a graph of $y = g(x)$ in Figure 24.

FIGURE 24



Example 15. Let $g(x) = -f(2(x+4)) + 3$. The original function $y = f(x)$ is shown in Figure 25. Explain how the graph of $y = g(x)$ is a transformation of the graph of the original function. Then sketch a graph of $y = g(x)$ in Figure 25.

FIGURE 25



Group Work. Complete the following for each set of functions below that your group is assigned:

- Identify and graph the basic function used in this transformation. (Example: $f(x) = x^2$). Use your Library of Functions Handout if necessary.
- State the series of transformations and the order in which they occur.
- Graph the transformation.
- Check your work. This can be done by hand by creating a table or with your graphing calculator.

Transformations

Section I: Horizontal and Vertical Shifts

(a) $g_1(x) = (x-5)^2 + 1$

(b) $g_2(x) = \sqrt{x+4} + 2$

(c) $g_3(x) = (x+1)^3 - 2$

(d) $g_4(x) = \dfrac{1}{x-2} + 3$

(e) $g_5(x) = |x+8| - 6$

(f) $g_6(x) = \sqrt[3]{x-4} - 2$

Section II: Horizontal and Vertical Stretches and Reflections

(a) $g_1(x) = \sqrt{-2x}$

(b) $g_2(x) = -5\sqrt[3]{x}$

(c) $g_3(x) = \left(-\frac{1}{2}x\right)^3$

(d) $g_4(x) = -\frac{1}{2}x^3$

(e) $g_5(x) = -3x^2$

(f) $g_6(x) = |5x|$

Section III: Combined Function Transformations

(a) $g_1(x) = 2\,|x| - 3$

(b) $g_2(x) = -(x+1)^3 - 3$

(c) $g_3(x) = \sqrt{-x} + 4$

(d) $g_4(x) = 3(x-2)^2 + 5$

(e) $g_5(x) = \dfrac{2}{x} + 5$

(f) $g_6(x) = 4\sqrt{2(x+1)} + 3$

FIGURE 26

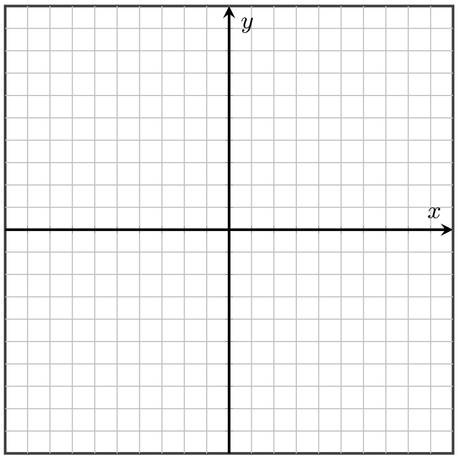


FIGURE 27

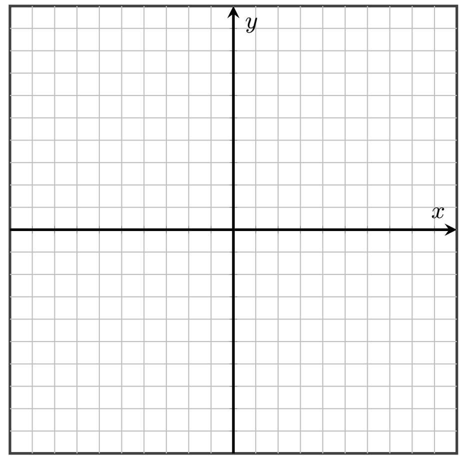


FIGURE 28

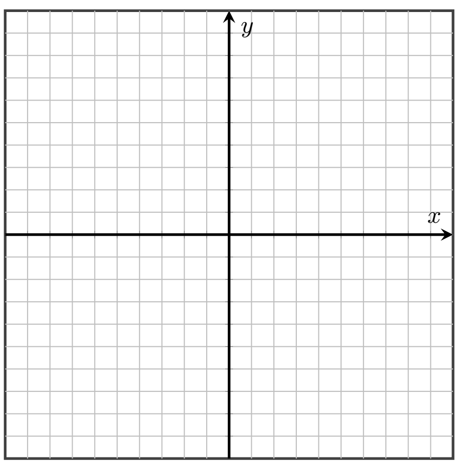


FIGURE 29

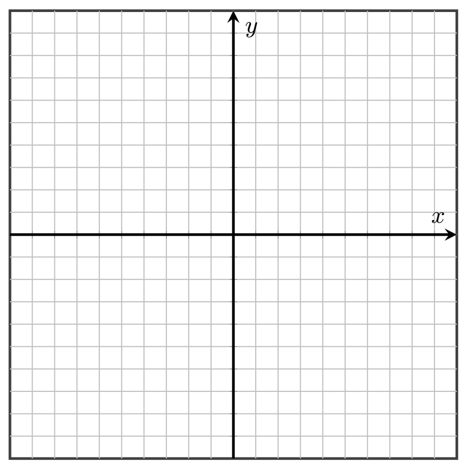


FIGURE 30

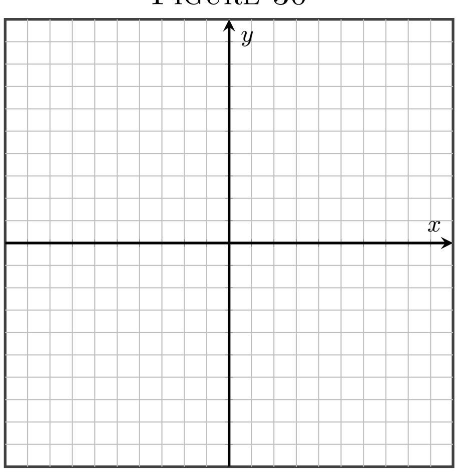


FIGURE 31