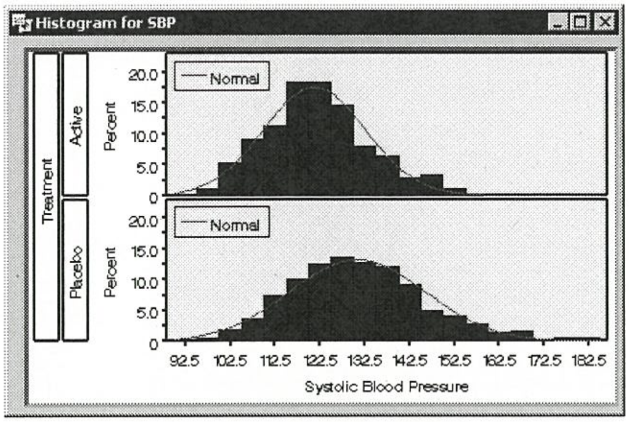
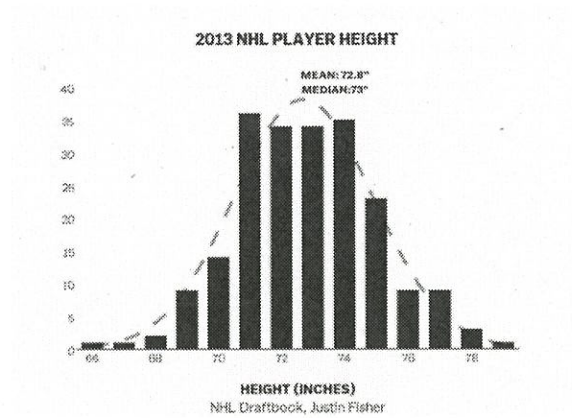
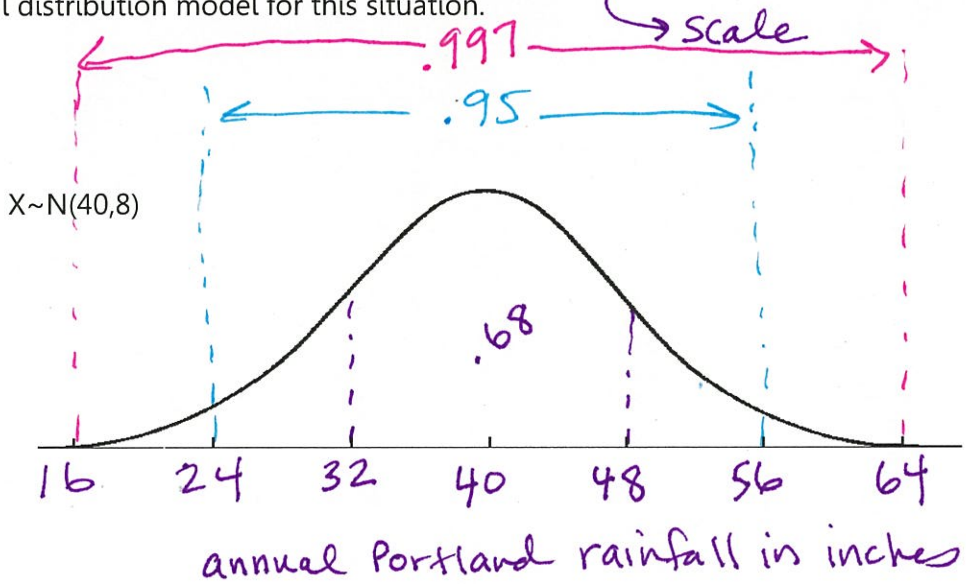
Here are some roughly symmetric, unimodal histograms





The Normal Model – The famous bell curve

Example 1. The mean annual rainfall in Portland is unimodal and approximately symmetric with a mean of 40 inches and a standard deviation of 8 inches, rounded to the nearest inch. Label the Normal distribution model for this situation.



Always label 3 standard deviations on each side of the mean (despite Geogebra's labeling)

68-95-99.7% Rule

In the normal model, about 68% of the values fall within 1 standard deviation of the mean, about 95% fall within 2 standard deviations of the mean, and about 99.7% fall within 3 standard deviations of the mean. This is also called the **Empirical Rule**. Label the bell curve above to show these key features.

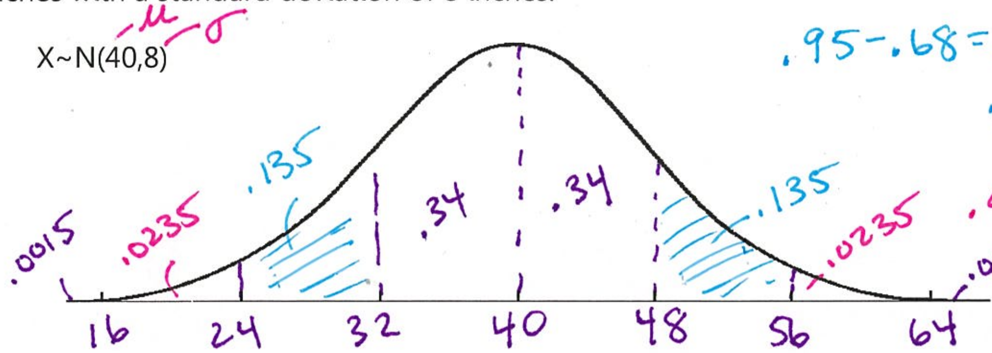
A value that is more than two standard deviations away from the mean is considered **unusual** or an **outlier**. A value that is more than three standard deviations away from the mean is **very rare**.

$X \sim B(n, p)$ $X \sim N(\mu, \sigma)$

Finding Probabilities using the Empirical Rule

Example 1 continued. To find a percentage using the Normal model, draw and label the model and shade the area or percentage that you want to find. In this case our mean annual rainfall in Portland is 40 inches with a standard deviation of 8 inches.

$X \sim N(\underset{\mu}{40}, \underset{\sigma}{8})$



$.95 - .68 = .27$

$\frac{.27}{2} = .135$

$.997 - .95 = .047$

$\frac{.047}{2} = .0235$

$1.0 - .997 = .003$

$\frac{.003}{2} = .0015$

a. What percentage of the time is the annual rainfall between 32 and 48 inches?

$P(\cancel{24}\, 32 \le X \le 48) = .68$ or 68%

b. What is the probability that the annual rainfall is more than 48 inches?

$P(X > 48) = .135 + .0235 + .0015$
$= .16$

c. What percentage of the time is the annual rainfall 24 inches or less?

$P(X \le 24) = .0235 + .0015 = .025$ or 2.5%

d. What is the probability that the annual rainfall is 56 inches or less?

$P(X \le 56) = .135 + .34 + .34 + .135 + .0235 + .0015$
$= .975$

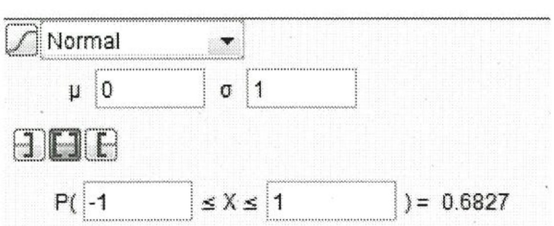
Finding Probabilities using GeoGebra

View > Probability Calculator > Select **Normal** in the dropdown menu under the graph

Type in the values for μ (the mean) and σ (the standard deviation)

Finding Normal Probabilities on GeoGebra

Select **]** for less than, **[ ]** for between two values, and **[** for greater than.



For a continuous distribution $>$ and $\geq$ are effectively the same

Activity: Find the exact probabilities for example 1 using GeoGebra.

a. $P(32 \leq X \leq 48) = .6827$ b. $P(X > 48) = .1587$

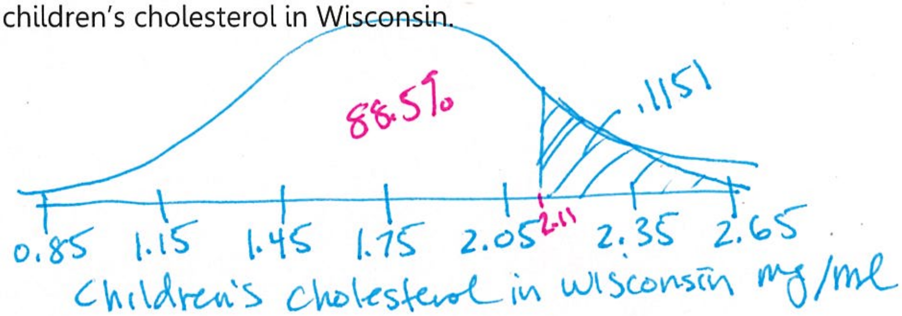
c. $P(X \leq 24) = .0228$ d. $P(X \leq 56) = .9772$

$<$ and $\leq$ are calculated the same way.

Practice. In a medical study the population of children in Wisconsin were found to have serum cholesterol levels that were normally distributed with a mean μ = 1.75 mg/ml and a standard deviation σ = 0.30 mg/ml.

a. Define and draw the Normal model for children's cholesterol in Wisconsin.

$X \sim N(1.75, .30)$



b. A child has a cholesterol level of 2.11 mg/ml. What is the percentage of children in Wisconsin who have cholesterol levels that are higher than this child's? Do you think their parents should be worried?

$P(X > 2.11) = .1151$ or 11.51% Only 11.5% are higher so I would be worried

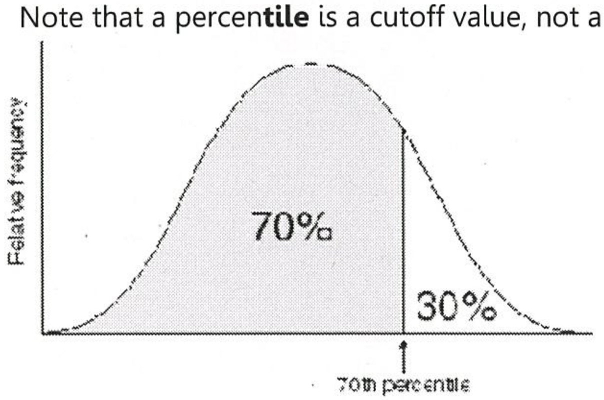
c. Find the percentage of children in Wisconsin who have cholesterol levels between 1.30 mg/ml and 2.23 mg/ml.

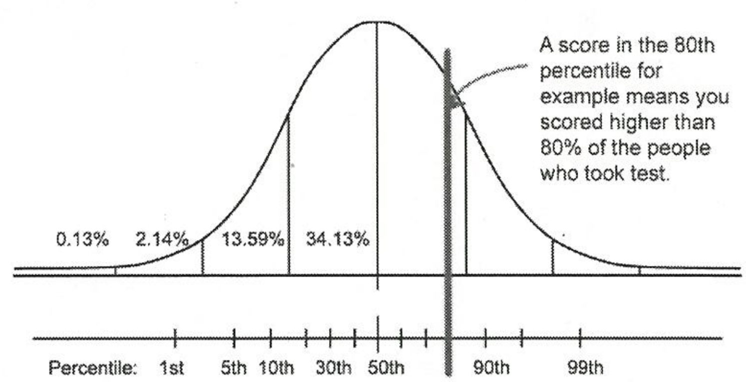
$P(1.30 \leq X \leq 2.23) = .8784$ or 87.84%

Percentiles and cutoff values

For any percentage of data, there is a corresponding **percentile** or **cutoff value.** That is the value that leaves the given percentage of data below it. We may be given a percentage and need to find the cutoff value or cut-score.

Note that a percen**tile** is a cutoff value, not a percent**age**.



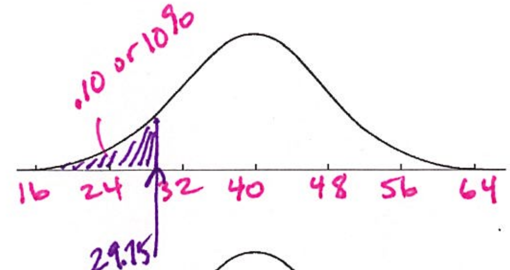


Example 2. Let's continue the rainfall example where the mean annual rainfall in Portland is 40 inches with a standard deviation of 8 inches. Shade and find the cutoff values for:

a. The lowest 10% of rainfall (the 10th percentile).

$P(X \leq \boxed{}) = .10$

29.7476

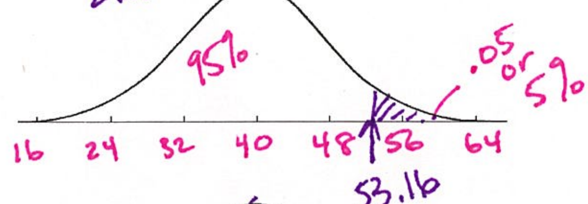


b. The highest 5% of rainfall (the 95th percentile).

$P(X \leq \boxed{}) = .95$

$P(X \geq \boxed{}) = .05$

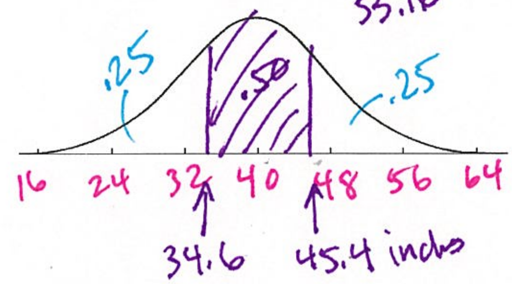
53.1588



c. The middle 50% of rainfall.

$P(X \leq \boxed{34.6041}) = .25$

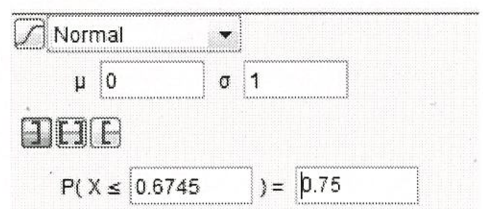
$P(X \leq \boxed{45.3959}) = .75$



Finding Inverse Normal Probabilities on GeoGebra

Type in the values for μ (the mean) and σ (the standard deviation)

Select **]** for less than or **[** for greater than. You can type in a probability and GeoGebra will calculate the cutoff value.

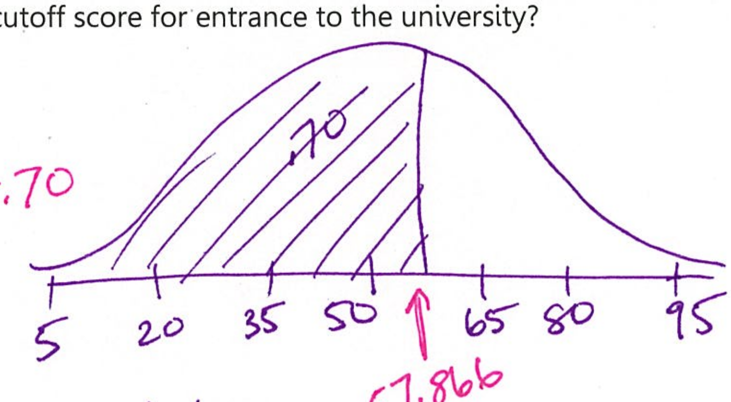


Example 3. Entry to a certain university is determined by a national test. The scores on this test are normally distributed with a mean of 50 points and a standard deviation of 15 points. Tom wants to be admitted to this university and he knows that he must score better than ~~at least~~ 70% of the students who took the test. What is the cutoff score for entrance to the university?

$X \sim N(50, 15)$

$P(X \leq \boxed{}) = .70$

57.866



He would need to score 57.866 or more on the test.

The Standard Normal Model, Z

Now we want to compare unlike events: Even if two events are quite different, we can still compare them using the standard deviation as a ruler. We can see how many standard deviations each event is away from its mean.

Example 4. Assume the average annual rainfall for in Portland is 40 inches per year with a standard deviation of 8 inches. Also assume that the average wind speed in Chicago is 10 mph with a standard deviation of 2 mph. Suppose that one year Portland's annual rainfall was only 24 inches and Chicago's average wind speed was 13 mph. Which of these events was more extraordinary?

Portland $X \sim N(40, 8)$ Chicago $X \sim N(10, 2)$

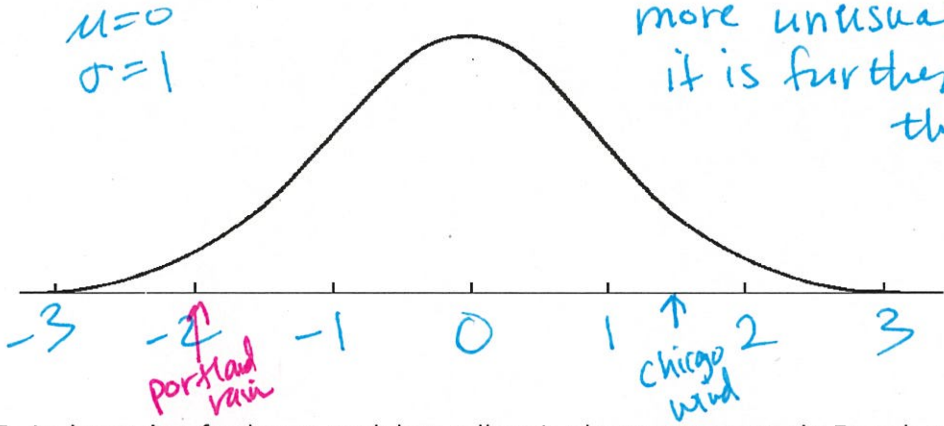
Z-score Formula: $Z = \frac{x-\mu}{\sigma}$

z-score for 24 inches of rain in Portland:	z-score for a wind speed of 13 mph in Chicago:
$z = \frac{24-40}{8} = -2$ 24 inches is 2 standard deviations below the mean	$z = \frac{13-10}{2} = 1.5$ 13 mph is 1.5 standard deviations above the mean

Standard Normal, Z~N(0,1)

$\mu = 0$
$\sigma = 1$

The rain in portland was more unusual because it is further from the mean



Example 5. An incoming freshman took her college's placement exams in French and mathematics. In French, she scored 82 and math 86. The overall results on the French exam had a mean of 72 points and a standard deviation of 8 points, while the mean math score was 68 points, with a standard deviation of 12 points. On which exam did she do better compared with the other freshman?

$F \sim N(72, 8)$ $M \sim N(68, 12)$

$z = \frac{82-72}{8} = 1.25$ standard deviations

$z = \frac{86-68}{12} = 1.5$ standard deviations

She scored better in math because she was more standard deviations above the mean.

Practice Problems

1. James has an adopted grandson whose birth family members are very short. After examining him at his 2-year checkup, the boy's pediatrician said that the z-score for his height relative to American 2-year olds was -1.88. Explain what that means.

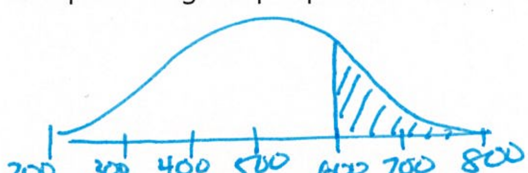
This boy's height is almost 2 standard deviations below the mean so that is quite short compared with all American 2-year-olds.

2. Assume a national math test score follows the normal model with mean μ = 500 and standard deviation σ = 100. We can symbolize this by saying: Test Scores ~ N(500, 100). Use this normal model to answer the following questions.

Draw a Normal sketch for each part. Write a probability statement and use GeoGebra to find each quantity.

Test Scores

a. The percentage of people who score 600 or higher.



$P(X \geq 600) = .1587$

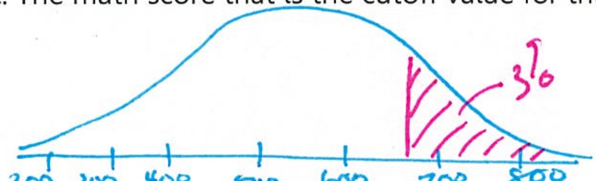
15.87% scored 600 or higher.

b. The probability that a person would score between 450 and 550 on the test.

$P(450 \leq X \leq 550) = .3829$

The chance of scoring between 450 and 550 is .3829.

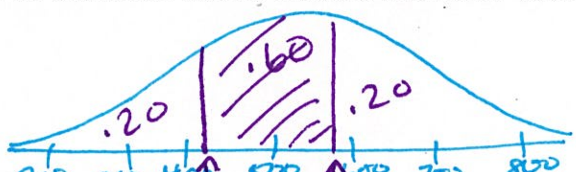
c. The math score that is the cutoff value for the highest 3% of scores.



$P(X \geq \boxed{688.0794}) = .03$

3% of scores were higher than 688.

d. The two math scores that are the cutoff values for the middle 60% of scores.

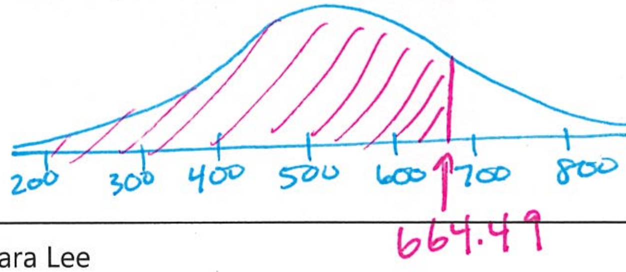


$P(X \leq \boxed{415.84}) = .20$

$P(X \geq \boxed{584.16}) = .20$

The middle 60% of scores were between 416 and 584.

e. What math score is the 95th Percentile?



$P(X \leq \boxed{664.49}) = .95$

The 95th percentile was 664 points.