Math 111 Lecture Notes

Section 4.2: Inverse Functions

Example 1. Temperature in degrees Fahrenheit, F, can be written as a function of temperature in degrees Celsius, C. This relationship is given by $F = g(C) = \frac{9}{5}C + 32$.

(a) Find and interpret $g(100)$.

(b) Solve and interpret the solution to $g(C) = 32$.

(c) Solve the equation $F = \frac{9}{5}C + 32$ for C.

A function f is said to be **one-to-one** if for every y-value in the range of f there is exactly one x-value in the domain of f.

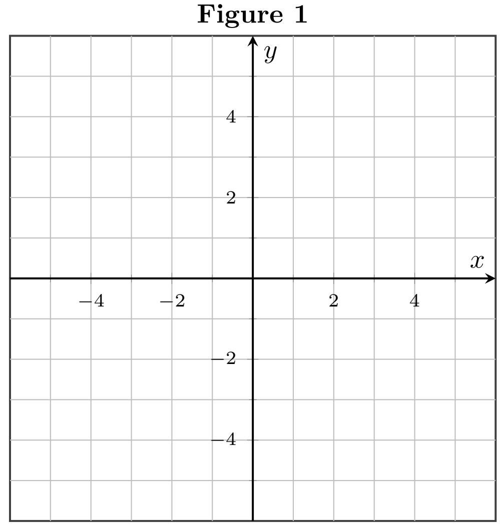
A function must be one-to-one in order to have an inverse. The **inverse function of** f reverses the process of the original function. In other words, the input and output switch roles. The original function is given by $y = f(x)$. The inverse function is given by $x = f^{-1}(y)$. If we want to graph both of these functions in the (x, y)-plane, then we use $y = f^{-1}(x)$.

The inverse function of f is denoted by f^{-1}. It is important to note that this notation *is not* denoting a reciprocal. That is, $f^{-1}(x) \neq \dfrac{1}{f(x)}$.

Example 2. Write the definition for $g^{-1}(F)$ for Example 1.

Example 3. The function f defined by $f(x) = 3x + 2$ is one-to-one. Find its inverse. Then graph $y = f(x)$ and $y = f^{-1}(x)$ in Figure 1. Include the graph of $y = x$ also.

Figure 1



Example 4. To verify that two functions are inverses, we show that $f\left(f^{-1}(x)\right) = x$ and that $f^{-1}\left(f(x)\right) = x$. Do this for the previous example.

Example 5. The function f defined by $f(x) = -\dfrac{2x}{x-1}$ is one-to-one. Find the inverse function. Confirm that the inverse function you found is correct by showing $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$.

State the domain and range of each f and f^{-1}.

The domain of f is the range of f^{-1}. Similarly, the range of f is the domain of f^{-1}.

The **horizontal line test** is a way of determining if a function is one-to-one. It states that if every horizontal line passes through a graph at most once, then the function is one-to-one.
In the same way that the vertical line test verifies if a graph represents a function, the horizontal line test verifies if the graph of a function is one-to-one (and thus invertible).

Example 6. Graph $y = f(x)$ for $f(x) = x^2$ in Figure 2. Then graph $y = g(x)$ for $g(x) = x^2, x \geq 0$ in Figure 3. Is either function invertible? Why or why not?

Figure 2. Graph of $y = f(x)$

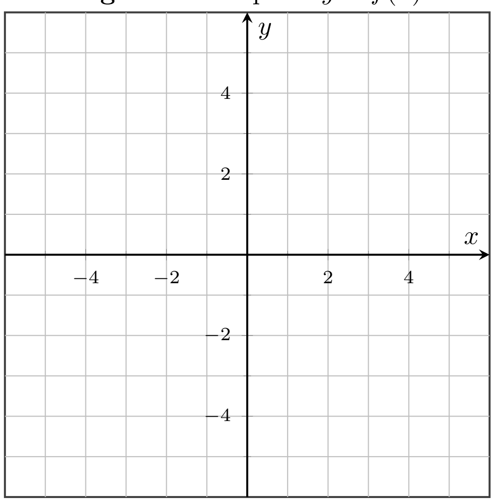
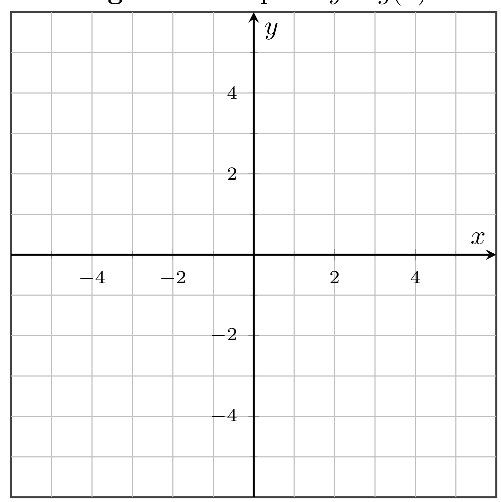
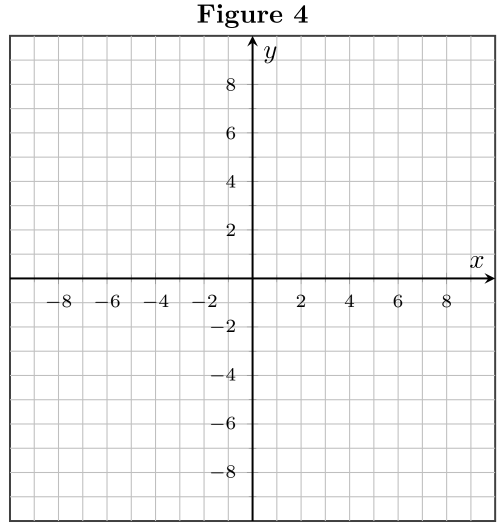


Figure 3. Graph of $y = g(x)$



Example 7. The function g defined by $g(x) = \sqrt[3]{x+8}$ is one-to-one. Find the inverse function and confirm that it is the inverse by showing $g\left(g^{-1}(x)\right) = x$ and $g^{-1}\left(g(x)\right) = x$. In Figure 4, use transformations to sketch $y = g(x)$, $y = g^{-1}(x)$ and $y = x$.

Figure 4



Example 8. Use the functions f and g given in Table 1 to determine the following.

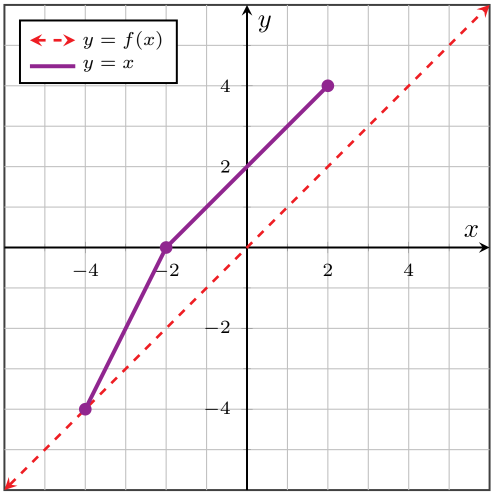
Table 1

x	-2	-1	0	1	2
$f(x)$	5	4	2	-1	1
$g(x)$	7	2	0	-2	9

(a) $g^{-1}(-2)$ (b) $f^{-1}(2)$ (c) $f^{-1}(0)$ (d) $f\big(g^{-1}(0)\big)$

Example 9. Graph the inverse function of f in Figure 5. Then use your sketch to find the values of f^{-1} below.

Figure 5



(a) $f^{-1}(-4)$ (c) $f^{-1}(0)$ (e) $f^{-1}(4)$

(b) $f^{-1}(-2)$ (d) $f^{-1}(2)$

Example 10. The diameter of a Window-Pane oyster, d (in mm), as a function of its weight, w (in grams) can be modeled by

$$d = f(w) = 25 + 20w^{1/3}$$

Find the inverse function by solving $d = 25 + 20w^{1/3}$ for w. Write this inverse function as $g(d)$.