Math 111 Lecture Notes

Section 4.5: Properties of Logarithms

Example 1. Calculate the following:

(a) $\log_5(1)$

(b) $\log_5(5)$

(c) $\log_2(1)$

(d) $\log_2(2)$

(e) $\log(1)$

(f) $\log(10)$

(g) $\ln(1)$

(h) $\ln(e)$

For any positive real number a, $a \neq 1$, it holds that

- $\log_a(1) = 0$
- $\log_a(a) = 1$

Example 2. We have said that the functions defined by $g(x) = \log_2(x)$ and $f(x) = 2^x$ are inverse functions. Find $f(g(x))$ and $g(f(x))$. Since f and g are inverses, what should these be equivalent to?

For any positive real numbers x and a, $a \neq 1$, it holds that

- $\log_a(a^x) = x$
- $a^{\log_a(x)} = x$

Example 3. Compare the following expressions:

$$\log_2(8) + \log_2(4) \qquad \text{vs.} \qquad \log_2(32)$$

Example 4. Compare the following expressions:

$$\log_3(81) - \log_3(3) \qquad \text{vs.} \qquad \log_3(27)$$

Example 5. Compare the following expressions:

$$4\log(10) \qquad \text{vs.} \qquad \log(10000)$$

For any positive real numbers M, N, and a, $a \neq 1$, it holds that

- $\log_a(MN) = \log_a(M) + \log_a(N)$
- $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$
- $\log_a(M^r) = r\log_a(M)$

Example 6. Use the properties of logarithms to find the exact value of the following expressions. Do not use a calculator.

(a) $\log_4\left(4^{-5}\right)$

(b) $\log_6(9) + \log_6(4)$

(c) $e^{\ln(7)}$

(d) $2^{\log_2(15)}$

(e) $\log(250) - \log(25)$

(f) $5^{\log_5(6)+\log_5(7)}$

Example 7. Write each expression as a sum and/or difference of logarithms. Express powers as factors.

(a) $\log\left(\dfrac{1}{x-3}\right), x > 3$

(b) $\ln\left(x^4\sqrt{1+x^2}\right)$

(c) $\log_5\left(\dfrac{\sqrt[3]{x^2+1}}{x^2-1}\right)$

Example 8. Write each expression as a single logarithm.

(a) $\log_2\left(\dfrac{x-3}{x+5}\right)+\log_2\left(\dfrac{3x+15}{x-4}\right)$

(b) $\log_4\left(\dfrac{5}{x}\right)-\log_4\left(\dfrac{x+2}{x^3}\right)$

(c) $\log\left(x^2+3x+2\right)-2\log\left(x+1\right)$

(d) $\ln\left(x^2-9\right)+\ln\left(\dfrac{x}{x-3}\right)-\ln\left(\dfrac{x+3}{x}\right)$

Change-of-Base Formula If a, b, and M are positive real numbers, $a \neq 1$ and $b \neq 1$, then

$$\log_a(M)=\frac{\log_b(M)}{\log_b(a)}$$

In practice, we primarily use one of the following forms of this formula:

$$\log_a(M)=\frac{\log(M)}{\log(a)} \quad \text{or} \quad \log_a(M)=\frac{\ln(M)}{\ln(a)}$$

Example 9. Use the Change-of-Base formula to write the following logarithmic expressions in terms of the natural logarithmic function or common logarithmic function. Then approximate each in your calculator.

(a) $\log_4(15)$

(b) $\log_5\left(\frac{1}{7}\right)$